Applications of Group Theory to Condensed Matter Physics: Course Notes

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674. Applications of Group Theory to Physics: Condensed Matter Physics. Theory of groups and group representations; point, space, and rotation groups; applications to molecular and crystal structures, crystal-field and spin-orbit interactions, energy bands and phonon dispersion relations. Applications to modern materials.

In contrast with common belief, this class is not really about group theory. The centerpiece of group theory, known as Galois theory, was developed in the 19th century by a brilliant French mathematician. He wrote down his theory the night before a duel and died early next morning at the tender age of twenty-something. Galois theory is usually considered very beautiful and aesthetic, at least by mathematicians. Actually, it is so beautiful that it bears no resemblance to the real world, and nobody I know could ever think of an application to physics. Group theory is one of the foundations of a mathematician's education, just like classical mechanics or Maxwell theory in physics. A good introduction to group theory is given in Serge Lang's book [1] on Algebra. It is tought at ISU in a mathematics course.

564. Theory of Groups. Commutators, p-groups, nilpotent groups, solvable groups, permutation groups, free groups, semidirect products, introduction to representation theory.

Instead, this physics course is about *representations of groups*, a subject developed in the 19th century by the German mathematician Emmy Noether. (Emmy Noether was the first woman to become a mathematics professor in Germany. She neede a lot of help from Gauss, I believe, to overcome the objections from her male colleagues. She did not receive a salary.) Mathematicians usually don't care much about representation theory, maybe because it is considered too simple or useless for other areas of mathematics.

When group theory was first applied to physical situations in the 1930s (mostly by Eugene Wigner), it was regarded by many as an unnecessarily complicated and almost perverse (German) tool of theoretical physics. Nowadays, however, group theory is seen as an essential component of every graduate (and even undergraduate) physics education. "Once a European esotericum, it has been supplanted by American practicalities", writes Brian Judd in the October 1994 issue of Physics Today.

I have not been able to find a good textbook on applications of group theory to condensed matter physics. While there are many good books containing the mathematical formalism [1] and other books dealing with applications of group representations to chemistry and physics, I could find no good synthesis of both. The textbook I chose two years ago [2] (Michael Tinkham's "Group Theory and Quantum Mechanics", McGraw-Hill, 1964) is very good, but 30 years old, and therefore lacks modern applications. (It also is very expensive.) Nevertheless, it is the *default* text for the group theory class at many universities, and has been used in the past at ISU. The book is on reserve in the physical science reading room and available at the book store. This time, I chose a different textbook [17] by S. J. Joshua. The first part of this book follows more or less the book by Tinkham (that's why I chose it), with some more modern material added. It still lacks the treatment of high-temperature superconductors. The second part of Joshua's books deals with magnetic groups, which we probably won't get into.

I am convinced that a modern condensed matter group theory course should be taught using a modern and concise mathematical language, and that is what I will try to achieve here. The first month or so I will spend on developing the mathematical formalism (groups, vector spaces, topology), the second month will deal with applications to classical and quantum mechanics, and the rest with examples from condensed matter physics. This schedule is roughly the same as in Tinkham's book, [2] but I will try to use a somewhat different (more mathematically precise) language and concentrate on more up-to-date examples, such as the energy band structure and phonon dispersion relations in semiconductors and 1-2-3 superconductors and related observable effects, such as optical absorption, Raman scattering, and electron-phonon interactions.

I intend this course to be an introduction into the scientific literature. I have prepared a script containing some of the mathematical formalism. The script is not necessarily designed for this class, but contains a lot of information. You may find it useful to refresh your memory and review some of the mathematics you have studied in the past. (Since I am not a mathematician, I cannot guarantee the accuracy of the material in the script, but I have done my best to collect the mathematics necessary for a course on the applications of group theory to condensed matter physics.) I have also copied a number of scientific articles on applications of group theory to semiconductor physics. These are also available on reserve in the link.

In order to make examples as clear as possible, I have made available a set of programs to calculate phonon and electron dispersion relations. The programs not only give you the energy (group theory tells us about degeneracies), but also the phonon eigenvectors and electronic wave functions. This will allow you to see in detail, what the symmetry properties and representations mean. The programs run on a VAX. Last time I taught the course, I used the ISU VAX computer, which has since been turned off. This time, I intend to use the Ames Lab VAX system. If any of you are not associated with Ames Laboratory, I will have to port the programs to Project Vincent.

While most of the groups we will deal with are finite, there is one important exception: The rotation group SO(3). Therefore, I will have to introduce and briefly discuss manifolds, Lie groups, and Lie algebras and make a short excursion into topology (compact spaces, connectedness, universal covering). This will help us to understand the projective representations needed in quantum mechanics because of the unkown phase factor and make the transition to SU(2) and introduction of the spin much more obvious.

1. SETS AND MAPPINGS

I assume that the reader is already familiar with sets and therefore avoid introducing them. As a matter of fact, it is non-trivial to define what a **set** is. For simplicity, we will not deal with sets that contain *themselves* as an element. This will keep us out of trouble. For completeness, I mention some of the commonly used notation.

If x is an object and M a set, then we write $x \in M$ (or $x \notin M$) if x is (or is not) an element of M. $M \cap N$ is the **intersection** of two sets M and N, $M \cup N$ their **union**. $M \cap N = \{x \mid x \in M \wedge x \in N\}$. $M \cup N = \{x \mid x \in M \vee x \in N\}$.

We say that N is a **subset** of M, $N \subseteq M$, if every element of N is also contained in M. Then, we can also call M a **superset** of N, $M \supseteq N$. The **difference** of two sets is defined as $M \setminus N = \{x \mid x \in M \wedge x \notin N\}$. If N is a subset of M, then the **complement of** N is $M \setminus N$.

Example:
The following sets are often used:
$\{\}$ or $\emptyset$ is the **empty set**.
$\mathbb{N}$ is the set of all **natural numbers**, i. e., the set of all positive integers. $\mathbb{Z}$ is the set of all **integers**, and $\mathbb{R}$ the set of all **real numbers**. $\mathbb{C}$ is the set all **complex numbers** and $\mathbb{Q}$ the set of all **rational numbers**, i. e., fractions.
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$; $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$;
$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$; $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

A **mapping** (or **map**) from a set M (called the **domain**) to a set N (called the **range**), i. e., $f : M \to N, x \mapsto f(x)$, is a rule which assigns a well-defined **image** $f(x) \in N$ to each **argument** $x \in M$. The **image of** f is defined as $\operatorname{Im}(f) = f(M) = \{f(x) \mid x \in M\}$. If M' is a subset of M, $M' \subseteq M$, then we can define the **image of** M', $f(M') = \{f(x) \mid x \in M'\}$. If N' is a subset of N, $N' \subseteq N$, then the **inverse image of** N' is given by $f^{-1}(N') = \{x \in M \mid f(x) \in N'\}$. Instead of mapping, we can also use the term **function**, in particular if M and/or N are sets of numbers. There is yet another term: If f is a map between sets of functions, then f is sometimes called an **operator**.

A mapping $f : M \to N, x \mapsto f(x)$ is called **injective** (or **one-to-one**), if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in M$. It is called **surjective** (or **onto**), if for each $y \in N$, there exists at least one $x \in M$ such that $f(x) = y$. We say that f is **bijective**, if it is both injective and surjective.

If $f : A \to B$ and $g : B \to C$ are maps, then we have a **composite map** $g \circ f$ such that $(g \circ f)(x) = g(f(x))$.

Let $f : A \to B$ be a map and $A' \subseteq A$ a subset of A. The **restriction** of f to A' is denoted as $f|A'$. Its graph consists of all pairs $(a, f(a))$ with $a \in A'$.

A set M is called **finite**, if it has a finite number of elements $|M|$. It is called **denumerable**, if there exists a bijective map from M into the set of natural numbers $\mathbb{N}$. M is called **countable**, if it is finite or denumerable. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are denumerable, but $\mathbb{R}$ and $\mathbb{C}$ are not denumerable. The number of elements in M is called the **cardinality** of M.

If M and N are sets, then we can define their **product** $M \times N$ as the set of all **pairs** (x, y)

with $x \in M$ and $y \in N$. As an example, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, and inductively $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. The same notation shall apply for other sets such as $\mathbb{C}$, $\mathbb{Z}$, etc. If we have maps $f : M \to N, x \mapsto f(x)$ and $f' : M' \to N', x' \mapsto f'(x')$, we can define the **product map** simply as $f \times f' : M \times M' \to N \times N', (x, x') \mapsto (f(x), f'(x'))$

There are two things we can do with sets: (i) We can define mappings $M \times M \to M$ with certain properties. This will yield operations such as sum and product and eventually lead to groups, rings, fields, and vector spaces. (ii) We can define some primitive geometrical characteristics. This is called *topology* and will be important when discussing continuous symmetry groups (translational and rotational symmetry) and the spin.

A class of sets with certain properties along with the maps that respect these properties (called **morphism**) is called a **category** by mathematicians. Such categories include groups, fields, vector spaces, topological spaces, manifolds, Lie groups, etc. We will see a lot of such categories in the rest of the text.

Exercises:

(X1) Let M and N be sets and $f : M \to N$ be a map. Show that f is bijective if and only if there is a map $g : N \to M$ such that the compositions $f \circ g$ and $g \circ f$ are the identities id_N and id_M of N and M.

2. EQUIVALENCE AND ORDERING RELATIONS

Let M and N be sets. Then $M \times N$ is the product of the two sets. A **relation** R between M and N is a subset of $M \times N$. We write $x \; R \; y$, if $(x, y) \in R$, but sometimes use a different symbol (for example $\sim, =, <, >, \leq, \geq$) instead of R. If $M = N$, we call R a relation in M.

Example:
A mapping $f : M \to N, x \mapsto f(x)$ is certainly a relation, since the **graph** of f, that is Graph $f = \{(x, f(x)) \mid x \in M\}$, is a subset of $M \times N$. But not every relation is also a mapping.

Definition:
A relation $R \subseteq M \times M$ is called
reflexive, if $x \; R \; x$ for all $x \in M$;
symmetric, if $x \; R \; y$ implies $y \; R \; x$ for all $x, y \in M$;
antisymmetric, if $x \; R \; y$ and $y \; R \; x$ implies $x = y$ for all $x, y \in M$;
transitive, if $x \; R \; y$ and $y \; R \; z$ implies $x \; R \; z$ for all $x, y, z \in M$.

Definition:
A relation $R \subseteq M \times M$ is called an **equivalence relation** if it is reflexive, symmetric, and transitive. We then write $x \sim y$ instead of $x \; R \; y$, if there are no ambiguties. For $x \in M$, we call the subset $[x] = \{y \in M \mid x \sim y\} \subseteq M$ the **equivalence class** of x.

Definition:
A relation $R \subseteq M \times M$ is called an **ordering relation** if it is reflexive, antisymmetric, and transitive. We then write $x \leq y$ instead of $x \; R \; y$, if there are no ambiguities. We write $x < y$, if $x \leq y$ and $x \neq y$. $y > x$ is supposed to mean the same as $x < y$ and $y \geq x$ the same as $x \leq y$ for all $x, y \in M$.

3. GROUPS AND HOMOMORPHISMS

How does a mathematician think of an algebraic operation, such as addition or multiplication ? For him this is a mapping $* : G \times G \to G, (x, y) \mapsto x * y$ (or **law of composition**) with certain properties. What does this imply ? In a physics textbook about groups, you will find the term **closure**, that is the product of two group elements is also a group element. This is implied by the definition of this mapping, since $(x, y) \mapsto x * y \in G$. We will simplify our notation by dropping the $*$ and simply write $(x, y) \mapsto xy$, unless there are any ambiguities. A simple way to define this mapping is using a **multiplication table**. (When writing down a multiplication table, try to arrange the rows and columns in such a way that the diagonal consists of all zeroes or ones.) Let us now define what a group is.

A **group** is a pair $(G, *)$ which consists of a set G and a law of composition $* : G \times G \to G, (x, y) \mapsto x * y$ with the following properties:
(G1) The **associative law** holds. That is $(xy)\,z = x\,(yz)$ for all $x, y, z \in G$.
(G2) There is a **neutral** (or **unit element**) e (also called 1 or 0 for multiplicative or additive notation) with $ex = xe = x$ for all elements $x \in G$.
(G3) Every element $x \in G$ has an **inverse element** x^{-1} (or $-x$ for additive notation) such that $xx^{-1} = x^{-1}x = e$.
If G is finite with h elements, then we call h the **order** of the group. We write $h = \text{Ord } G$ or $h = |G|$.

Definition: A group G is called **commutative** (or **Abelian**), if for all $x, y \in G$, we have $xy = yx$. For such groups, we will normally reserve the addititive notation.

Definition: Let G, H be groups with neutral elements e and e'. A mapping $f : G \to H$ is called a **homomorphism** (of groups) if $f\,(xy) = f\,(x)\,f\,(y)$ for all $x, y \in G$ and $f\,(e) = e'$. If $G = H$, we call f an **endomorphism**. If f is surjective (onto), then we say that f is an **epimorphism**. If f is injective (one-to-one), then f is called a **monomorphism** or an **embedding**. If $e' \in H$ is the neutral element, we call the subset $\text{Ker} f = f^{-1}\,(e') \subseteq G$ the **kernel** of f. We call f an **isomorphism**, if there is a map $f' : H \to G$ such that the compositions $f \circ f'$ and $f' \circ f$ are the identities in G and H. An isomorphism is bijective. An isomorphism $f : G \to H$ with $G = H$ is called **automorphism**.

Example:
The set of all integral numbers $\mathbb{Z}$ is a denumerable group, together with addition. It is not a group under multiplication, since 0 does not have an inverse element. Division by 0 is not defined. This group $(\mathbb{Z}, +)$ is Abelian. The natural numbers $\mathbb{N}$ do not form a group under addition, since there are no inverse elements.
As an example of a non-Abelian group, consider the set $\text{GL}\,(n, \mathbb{R})$ (or $\text{GL}\,(n, \mathbb{C})$) of all $n \times n$ matrices with real (or compex) coefficients and non-vanishing determinants. The law of composition in this case is matrix multiplication.
In this course on representations of groups to solid state physics, we will mostly deal with symmetry groups, that is the set of all symmetry operations (such as reflection, rotation, or inversion) which would bring an object (such as a molecule or crystal) into a position indistinguishable from the original.

Definition: A group G is called a **cyclic group**, if there exists an element $a \in G$ (called **generator**) which generates G, i.e., $G = \{a^n \,|\, n \in \mathbb{Z}\}$. (This set is not necessarily infinite, since the series may be periodic.) We see that a finite group G with n elements is a cyclic group, if G contains an element $a \in G$ (the generator) such that $G = \{a, a^2, \ldots, a^n\}$.

Example:
Let G and H be groups. Then we define the **direct product** (of groups) as the direct product $G \times H$ of sets and the multiplication as $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ for $g_1, g_2 \in G; h_1, h_2 \in H$. Apparently, (g^{-1}, h^{-1}) is the inverse of (g, h) for $g \in G, h \in H$.

Example:
Consider the following finite groups: The **symmetric group** (or **permutation group**) $S(n)$ is the group of all permutations of the numbers $\{1, \ldots, n\}$. Every permutation can be written as a sequence of **transpositions** (switching only two elements leaving the others fixed), therefore it makes sense to talk about even and odd permutations, depending on the number of transpositions. (This is well-defined.) The **alternating group** $A(n)$ is the set of all even permutations of $\{1, \ldots, n\}$. $A(n)$ is a normal subgroup within $S(n)$.

Example:
Periodic boundary conditions: For $n \in \mathbb{N}$, consider the set $G = \{0, 1, \cdots, n-1\} \subseteq \mathbb{N}$. For $a, b \in G$, we define a law of composition $\oplus$ as follows: We set $a \oplus b = c$, if there are $i \in \mathbb{Z}, c \in G$ with $a + b = in + c$, where $+$ is the normal addition of numbers in $\mathbb{Z}$ (division with remainder). This law of composition defines an Abelian cyclical group denoted as $\mathbb{Z}/\mathbb{Z}n$, in other words this is the factor group of cosets $a\mathbb{Z}$ in $\mathbb{Z}$. We immediately simplify our notation and write the addition in $\mathbb{Z}/\mathbb{Z}n$ as $+$. The physical significance of this definition is the following: Take a one-dimensional linear chain of atoms evenly spaced. Clearly, this has some resemblance to the set of integers $\mathbb{Z}$. Now select a substring of $n+1$ adjacent atoms, cut off everything to the left and right, and identify the two atoms on the edges. This is what is called periodic boundary conditions.

Example:
Cyclic groups of order n: The complex numbers $z_n = \exp(i2\pi/n)$, together with complex multiplication, form a cyclic group of order n.

Exercises:

(X1) Show that there is only one neutral element in each group.

(X2) Show that for each element $x \in G$, there is only one inverse x^{-1}.

(X3) Is there a group with zero elements ?

(X4) Let G be a cyclic group with n elements and the neutral element e. Show that this group is Abelian. Show that $a^n = e$ for all $a \in G$.

(X5) Show that $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

(X6) How many elements are in the set $G = \{a^n \,|\, n \in \mathbb{Z}\}$?

4. SUBGROUPS, COSETS, AND QUOTIENT GROUPS

Definition: For a group G, a **subgroup** H of G is a subset $H \subseteq G$ which is closed under multiplication (or addition) and inverse, i.e., $xy \in H$ and $x^{-1} \in H$ for all $x, y \in H$. Trivial examples for subgroups are the whole group G and the group only consisting of the neutral element. The subgroup $\{e\}$ is called the **trivial** subgroup.

Theorem:
(a) The image $f(G)$ of a homomorphism $f : G \to H$ is a subgroup of H.
(b) The kernel $\mathrm{Ker} f$ of a homomorphism f is a subgroup of G.

Definition:
Let G be a group and $H \subseteq G$ a subgroup of G. We define the **left coset** as $aH = \{ah \mid h \in H\}$ and the **right coset** as $Ha = \{ha \mid h \in H\}$ for $a \in G$. The set of all left cosets of H in G is denoted as G/H, the set of all right cosets as $G\backslash H$. For $a \in G$ and a left coset aH, we obtain a right coset Ha^{-1}. This mapping $\Phi : G/H \to H\backslash G : aH \mapsto Ha^{-1}$ is bijective, therefore we can limit our discussions to left cosets only. In particular, the number of elements in G/H is the same as the number of elements of $G\backslash H$. This number is called the **index** of H in G and denoted as $[G : H]$ or $\mathrm{Ind}_G H$. H is called **normal** (or a **normal divisor** or an **invariant subgroup**), if $aH = Ha$ for all $a \in G$. In this case, we can simply call $aH = Ha$ a **coset**.

Remark: Every subgroup N of index $[G : N] = 2$ is normal.

Definition:
Let G be a group and $H \subseteq G$ a subgroup of G. We can call two group elements $a, b \in G$ **equivalent**, if there is an $h \in H$ such that a=bh. This condition defines an **equivalence relation**. The equivalence class of a is denoted as $[a] = aH$. We see immediately that two left cosets are either identical or have no element in common (transitivity).

Theorem (Lagrange):
If H is a subgroup of a group G, then $|G| = [G : H] \cdot |H|$. In other words, the order of a subgroup divides the order of the group. We can conclude that a group G whose order is a prime has only the trivial subgroups G and $\{e\}$. Therefore, every element $g \in G, g \neq e$ generates the group. That implies that G is cyclic and Abelian. (See Tinkham, page 10, for a proof of this theorem.)

Definition:
Let N be a normal divisor of the group G. There is one and only one law of composition on the set G/N of all cosets of N in G, such that the canonical projection $\pi : G \to G/N, g \mapsto gN$ is a homomorphism. G/N is a group with this law of composition (called the **quotient group** or **factor group**), and **Ker** $\pi = N$. The order of G/N is $|G/N| = [G : N] = |G| / |N|$.

Exercises:

(X1) Show that the kernel $\mathrm{Ker} f$ of a homomorphism $f : G \to H$ is a subgroup of G. Show that for each subgroup $G' \subseteq G$ the image $\mathrm{Im} f = f(G')$ is a subgroup of H.

(X2) Show that if the kernel of a homomorphism is trivial (i.e., consists of only the neutral element), then the homomorphism is injective.

(X3) Write down the multiplication tables for the set $\mathbb{Z}/\mathbb{Z}5$ with addition and multiplication as the laws of composition. Make sure that the diagonal of the tables all consist of zeroes (for addition) and ones (for multiplication). Are they groups ? What about $\mathbb{Z}/\mathbb{Z}4$? Are these groups also ? Now study the sets $\mathbb{Z}/\mathbb{Z}4 \setminus \{0\}$ and $\mathbb{Z}/\mathbb{Z}5 \setminus \{0\}$ with multiplication as the law of composition.

(X4) Find all groups with one, two, three, and four elements and write down their multiplication tables. Which of these groups are Abelian ? (Hint: You may want to use Lagrange's theorem.)

5. CONJUGATION CLASSES

Let G be a group and $x \in G$. We define $\sigma_x : G \to G$ to be the map such that $\sigma_x(y) = xyx^{-1}$ for all $y \in G$. This map is called **conjugation**. It is an automorphism of G, since $\sigma_x(yz) = \sigma_x(y)\,\sigma_x(z)$, with $\sigma_{x^{-1}}$ being the inverse mapping of σ_x.

If we denote the **group of automorphisms** of G as $\mathrm{Aut}(G)$, then $\sigma : G \to \mathrm{Aut}(G), x \mapsto \sigma_x$ defines a homomorphism of G into its group of a automorphisms. The kernel of this homomorphism is a normal subgroup of G, which consists of all $x \in G$ such that $xyx^{-1} = y$ for all $y \in G$, i.e., all $x \in G$ which commute with every element of G. This kernel is called the **center** of G.

Two elements $x, y \in G$ are called **conjugate**, if there is some $z \in G$ with $x = zyz^{-1}$. Being conjugate defines an equivalence relation in G, since it is obviously symmetric, reflexive, and transitive. As usual, we denote the **conjugacy classes** as $[x]$ for all $x \in G$. The number of conjugacy classes of a group G is called the **class number**.

Two subsets A and B of G are called **conjugate**, if there is some $x \in G$, such that $B = xAx^{-1} = \{xax^{-1} \mid a \in A\}$.

We see that being a normal divisor H in a group G implies $aHa^{-1} = H$ for all $a \in G$. Therefore, if h is in H, all elements conjugate to h are also in H. In other words, a normal divisor is made up of complete classes.

The conjugacy classes of the direct product of two groups G and H are the pairs of all the conjugacy classes in G and H. In other words, (g, h) and (g', h') in $G \times H$ belong to the same class if and only if g belongs to the same class as g' in G and h to the same class as h' in H. If G has n classes and H has m classes, the $G \times H$ has nm classes.

Exercises:

(X1) Let G be a group. Show that the set $\mathrm{Aut}(G)$ of automorphisms of G is a group. What about the set $\mathrm{End}(G)$ of all endomorphisms of G ?

(X2) Show that the set of all conjugacy classes of a group G is also a group.

6. RINGS AND FIELDS

A **ring** A is a set with *two* laws of composition called multiplication and addition, respectively, with the following properties:
(R1) With respect to addition $(A, +)$, A is a commutative group with the neutral element 0.

(R2) The multiplication is associative and has a neutral element 1.
(R3) For all $x, y, z \in A$, we have the **distributivity** law:
$(x + y)\, z = xz + yz$ and $z\,(x + y) = zx + zy$.
The ring A is called **commutative**, if $ab = ba$ for all $a, b \in A$. An element $a \in A$ is called a **unit**, if there exists an element $b \in A$ with $ab = ba = 1$. A commutative ring A is called a **field**, if $1 \neq 0$ and every element $a \in A, a \neq 0$ is a unit, i.e., has an inverse element a^{-1} with $aa^{-1} = a^{-1}a = 1$. In other words, a commutative ring A is a field, if $(A \setminus \{0\}, \cdot)$ is an Abelian group.

Example:
(E1) The integral numbers form a commutative ring $(\mathbb{Z}, +, \cdot)$. 1 and -1 are the only units in $\mathbb{Z}$.
(E2) The classical examples for fields are the sets of numbers $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.
(E3) $(\mathbb{Z}/\mathbb{Z}n, +, \cdot)$ is a commutative ring. It is a field if and only if n is a prime.

Definition (Special ring elements):
Let A be a commutative ring and $a \in A$.
a is called a **divisor of zero** if there is an element $b \in A$ such that $ab = 0$.
a is called **nilpotent**, if there is an $n \in \mathbb{N}$ with $a^n = 0$.
a is called **idempotent**, if $a^2 = a$.
a is called **unipotent**, if $1 - a$ is nilpotent.

Exercises:

(X1) Let $(A, +, \cdot)$ be a ring with the additive neutral element 0 and the multiplicative neutral element 1. Show that for $a, b \in A$ and $m, n \in \mathbb{Z}$: (Do NOT assume that A is commutative.)
(a) $a \cdot 0 = 0 \cdot a = 0$.
(b) $a\,(-b) = (-a)\, b = -\,(ab)$.
(c) $(ma)\,(nb) = (mn)\,(ab)$.

(X2) Find the nilpotent, idempotent, and unipotent elements in the ring $\mathbb{Z}/\mathbb{Z}4$. What are the units and divisors of zero ? Hint: It may help to set up a multiplication table of $(\mathbb{Z}/\mathbb{Z}4, \cdot)$.

7. MODULES AND VECTOR SPACES

I would love to immediately introduce vector spaces and avoid to define what a module is. But unfortunately, a Bravais lattice is a module, not a vecctor space. This chapter defines the term **action**. This is a very important concept and will show up again when we define the representation of a group. Do not mix up action and law of composition.

Definition:
Let M and X be sets. An **action** of M on X is a mapping $M \times X \to X$.

Definition:
Let $(A, +, \cdot)$ be a commutative ring and $(V, +)$ an Abelian group. We say that V is an A-module (or module over A or simply a **module**), if there is an action $A \times X \to X, (a, x) \mapsto a \cdot x$ with the following properties: For all $a, b \in A$ and $x, y \in V$ we have:
(M1) $1_A \cdot x = x$;
(M2) $a\,(bx) = (ab)\, x$;
(M3) $a\,(x + y) = ax + ay$;

(M4) $(a+b)x = ax + bx$.
The elements of V are called **vectors**, the elements of A **scalars**. V is called a **vector space**, if A is a field.

Example:
(E1) The prototypes of vector spaces are the real vector space $\mathbb{R}^n$ (over the field $A = \mathbb{R}$) and the complex vector space $\mathbb{C}^n$ (over the field $A = \mathbb{C}$) for $n \in \mathbb{N}$. We have an example for a module, if we only allow integral coordinates. In other words, $\mathbb{Z}^n$ is a module.
(E2) A commutative ring A is an A-module over itself.
(E3) Any commutative group is a $\mathbb{Z}$-module.
(E4) The trivial group $\{e\}$ is an A-module over any commutative ring A.

Definition:
Let A be a commutative ring and V an A-module. A subset $U \subseteq V$ is called a **submodule** (or **subspace** if A is a field) of V if
(1) U is not empty.
(2) $x + y \in U$ for all $x, y \in U$.
(3) $ax \in U$ for all $x \in U$, $a \in A$.
It is easy to see that the intersection of two submodules is also a submodule.

Definition:
An A-module V is said to be simple if it does not contain any submodule other than 0 and V itself, and if $V \neq 0$.

Definition:
Let A be a commutative ring and V, W be A-modules. Let $f : V \to W$ be a mapping. f is called a **homomorphism** (of modules) or an A-**linear** mapping if f is a homomorphism of groups and satisfies $f(ax) = af(x)$ for all $a \in A, x \in V$. The kernel of f is a submodule in V and the image of f a submodule of W. If f is bijective, then we call it an **isomorphism**, and V and W are said to be **isomorphic**.

Example:
Let V be an A-module and $U \subseteq V$ be a submodule. Then we can define the **factor module** V/U as V/U of groups and the induced product between scalars and vectors. The canonical projection $\pi : V \to V/U$ is A-linear.

Example:
For two A-modules V and W, we can define the **direct product** $V \times W$ (of modules) as their product $V \times W$ of groups and $a(v, w) = (av, aw)$ for $a \in A, v \in V, w \in W$. This construction is sometimes also called the **direct sum** $V \oplus W$. (The direct sum and direct product of an infinite family of modules $(V_i)_{i \in I}$ are not the same. The direct sum consists of all those I-tuples $(x_i)_{i \in I}$ with almost all coefficients x_i equal to zero.) If we have two A-linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, we can define the **direct sum of homomorphisms** $f_1 \oplus f_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$, $(v, w) \mapsto (f_1(v), f_2(w))$. (The direct sum and the **direct product** of a finite number of homomorphisms are the same. An infinite product and an infinite sum of homomorphisms is not the same, as described above.) For representations of groups in condensed matter physics, it is customary to use the direct sum notation.

Example:
Sum of submodules: Let V be an A-module over a commutative ring A and $U_i, i \in I$ a family of submodules of V. Then the submodule U generated by $\cup_{i\in I} U_i$ consists of all elements of the form $\sum_{i\in I} x_i$ with $x_i \in U_i$ with almost all x_i equal to zero. This sum of submodules is denoted as $U = \sum_{i\in I} U_i$.

Exercises:
(X1) Let V and W be A-modules and $f : V \to W$ be a homomorphism. Show that $\mathrm{ker} f$ is a submodule in V and $\mathrm{Im} f$ a submodule in W.

8. LINEAR COMBINATIONS AND BASES

Definition:
Let V be an A-module over a commutative ring A and $S \subseteq V$ a subset of V. By a **linear combination** of elements of S (with coefficients in A) one means a sum

$$\sum_{x\in S} a_x x$$

where **the coefficients** $\{a_x\}_{x\in S}$ are a set of elements of A, almost all of which are zero. (**Almost all** in mathematics means all but a finite number.) The set of all linear combinations of elements of S is a submodule U in V and called the submodule **generated by** S. S is called a **set of generators** for U. A module V is called **finitely generated** or **finite** over A, if there exists some finite subset $S \in V$ that generates V.

Definition:
Let V be an A-module over a commutative ring A and $S \subseteq V$ a subset of V. The elements of S are called **linearly independent** (over A) if whenever we have a linear combination

$$\sum_{x\in S} a_x x = 0,$$

then $a_x = 0$ for all $x \in S$. Otherwise, S is called **linearly dependent**.

Definition:
Let V be an A-module over a commutative ring A and $S \subseteq V$ a subset of V. We shall say that S is a **basis** of V if S is not empty, if S generates V, and if S is linearly independent. A **free** module is a module which admits a basis, or the zero module.

Example:
If we have two A-modules V and W with bases $S = \{x_i\}_{i\in I}$ of V and $T = \{y_i\}_{i\in I}$ of W, then there exists a unique isomorphism $f : V \to W$ such that $f(x_i) = y_i$ for all $i \in I$.

A module not necessarily has a basis. Even if we have two bases for the same module, then the two bases not necessarily have the same number of elements. For vector spaces, things tend to be easier, because we can divide by coefficients.

Theorem:
Let V be a finitely generated vector space over a field K and assume that $V \neq \{0\}$.

(i) V has a basis.
(ii) If Γ_1 and Γ_2 are bases, then they have the same cardinality. This cardinality is called the **dimension** $\dim_K V$ of V over K.
(iii) Let Γ be a set of generators of V. Then there is a subset of Γ which is a basis.
(iv) Let Γ be a set of linearly independent elements of V. Then we can add other elements of V to Γ such that this new set will be a basis.
(v) Let $U \subseteq V$ be a subspace of V. Then $\dim_K V = \dim_K U + \dim_K V/U$.
(vi) If W is also a K-vector space and $f : V \to W$ linear, then $\dim_K V = \dim_K \mathrm{Ker} f + \dim_K \mathrm{Im} f$.

9. TENSOR PRODUCTS

Tensor products are some of most abstract (and, to many, unmotivated) constructions in linear algebra. Unfortunately, they are very important for the theory of representations of groups (selection rules) and we cannot do without them.

Definition: For a commutative ring A, let V and W be A-modules with A-bases $(v_i)_{i\in I}$ of V and $(w_j)_{j\in J}$ of W. Then the family $(v_i \otimes w_j)_{i\in I, j\in J}$ forms a basis of the **tensor product** $V \otimes W$. The operation of A on $V \otimes W$ follows from the fact that the map

$$V \times W \to V \otimes W, \quad (v, w) \mapsto v \otimes w \tag{9.1}$$

is supposed to be A-bilinear for all $v \in V$ and $w \in W$. This means that

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \tag{9.2}$$
$$(ax \otimes y) = (x \otimes ay) = a\,(x \otimes y) \tag{9.3}$$

for all $x_1, x_2, x \in V$, $y \in W$, and $a \in A$. This definition can inductively be extended to allow for tensor products of more than two modules.

If this definition does not make you too happy, refer to any good book on abstract linear algebra (for example Serge Lang's algebra book [1]) for a precise definition and examples.

Warning: Elements of the form $x \otimes y$ generate the tensor product $V \otimes W$, but not every element in $V \otimes W$ can be written in this form.

The tensor product is associative and commutative in the following sense:

Theorem: Let V_1, V_2, V_3 be A-modules. Then there exists a unique isomorphism

$$(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3) \tag{9.4}$$

such that

$$(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \tag{9.5}$$

for all $x \in V_1, y \in V_2$ and $z \in V_3$.

Theorem: Let V and W be A-modules. Then there is a unique isomorphism $V \otimes W \to W \otimes V$ such that $x \otimes y \mapsto y \otimes x$ for $x \in V$ and $y \in W$.

Theorem: The tensor product also is distributive. For three A-modules U, V, and W, there is a canonical isomorphism

$$U \otimes (V \oplus W) \;\cong\; (U \otimes V) \oplus (U \otimes W)\,. \tag{9.6}$$

Definition: If we have two A-linear maps $f_1 : V_1 \to W_1$ and $f_2 : V_2 \to W_2$, we can define the **tensor product of homomorphisms** as $f_1 \otimes f_2 : V_1 \otimes V_2 \to W_1 \otimes W_2$, $v \otimes w \mapsto f_1(v) \otimes f_2(w)$.

Sometimes, we will have to deal with more exotic constructions from linear algebra, for example the **exterior** or **wedge** product. Have you ever seen this:

$$dV \;=\; dx \wedge dy \wedge dz. \tag{9.7}$$

Let me give you a **physical motivation** for the tensor product: Assume that you have a quantum-mechanical system consisting of two particles which do not interact. The wave function $\Psi(\vec{r}_1, \vec{r}_2)$ for the complete system is then given by the product $\phi(\vec{r}_1)\,\psi(\vec{r}_2)$ of the individual wave functions $\phi(\vec{r})$ and $\psi(\vec{r})$ of the two particles. If V is the Hilbert space containing all wave functions of the first particle, and W the Hilbert space containing all wave functions of the second particle, then the Hilbert space of all wave functions for the two-particle system is given by $V \otimes W$. Why is that ? You will see that the product $\phi\psi$ of the wave functions (as complex numbers) has the same bilinearity properties as the tensor product, therefore the two constructions are identical. Well yes, there are other bilinear entities in linear algebra, but these have further properties which the product of two functions does not have. For example, there are Slater-Determinants for Fermions and other constructions for Bosons with corresponding mathematical concepts.

10. SCALAR PRODUCTS, DUAL SPACES, AND TRACE

For each K-vector space V, there is a **scalar product** (if $K = \mathbb{R}$) or a **Hermitian product** (if $K = \mathbb{C}$), that is a map

$$V \times V \mapsto K, \quad (v, w) \mapsto \langle v, w\rangle \tag{10.1}$$

with certain properties that I assume you already know.

Definition: Just a reminder from linear algebra: Every finite-dimensional vector space V is canonically isomorphic to its **dual space** $V^* = \mathrm{Hom}(V, K)$, that is the dual space consists of linear functions from V into K. The isomorphism uses the inner (scalar or Hermitian) product and goes like this:

$$v \mapsto v^* = \langle v, -\rangle = (u \mapsto \langle v, u\rangle) \tag{10.2}$$

.

Of particular importance for representation theory is the isomorphism

$$\theta : V^* \bigotimes V \to \mathrm{Hom}\,(V,V)\,, \tag{10.3}$$

$$v^* \otimes w \mapsto (u \mapsto v^*\,(u)\,w)\,. \tag{10.4}$$

If $f \in \mathrm{Hom}\,(V,V)$ is a K-linear enomorphism of V, then the map

$$Tr : \mathrm{Hom}\,(V,V) \cong V^* \bigotimes V \to K \tag{10.5}$$

$$v^* \otimes u \mapsto v^*\,(u) \tag{10.6}$$

associates to $f \in \mathrm{Hom}\,(V,V)$ its **trace** $\mathrm{Tr}\,(f) \in K$. If $v_1, \ldots, v_n$ is a basis of V and $f\,(v_j) = \sum_i a_{ij} v_i$, then $\theta\,(\sum a_{ik} v_k^* \otimes v_i) = f$. Consequently,

$$\mathrm{Tr}\,(f) \;=\; \sum_i a_{ii}, \tag{10.7}$$

or in other words: The trace is the sum of the diagonal elements of the matrix associated with f.

Proposition: Properties of the trace:

(i) $\mathrm{Tr} : \mathrm{Hom}\,(V,V) \to K$ is linear.
(ii) $\mathrm{Tr}\,(\phi f \phi^{-1}) \;=\; \mathrm{Tr}\,(f)$ for each K-automorphism ϕ of V.
(iii) For $f : V \to W$ and $h : W \to V$, $\mathrm{Tr}\,(fh) = \mathrm{Tr}\,(hf)$.
(iv) $\mathrm{Tr}\,(f \oplus h) = \mathrm{Tr}\,(f) + \mathrm{Tr}\,(h)$.
(v) $\mathrm{Tr}\,(f \otimes h) = \mathrm{Tr}\,(f) \cdot \mathrm{Tr}\,(h)$.
(vi) $f : V \to V$ induces a map $f^* : V^* \to V^*$ and $\mathrm{Tr}\,(f^*) = \mathrm{Tr}\,(f)$.
(vii) If $f : V \to V$ is idempotent (that is $f^2 = f$), then $\mathrm{Tr}\,(f)$ is the dimension of the image of f.
(viii) $\mathrm{Tr}\,(A^{-1}MA) = \mathrm{Tr}\,(M)$ for all $A, M \in \mathrm{GL}\,(n,K)$.

11. BRAVAIS LATTICES

In this paragraph, the commutative ring A will be $A = \mathbb{Z}$, the ring of integral numbers.

A **Bravais lattice** V is a $\mathbb{Z}$-module that is a subset of $\mathbb{R}^3$ and isomorphic to the $\mathbb{Z}$-module $\mathbb{Z}^3$, with $f : V \to \mathbb{Z}^3$ being the isomophism. If $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, and $\vec{k} = (0,0,1)$ is the canonical basis of $\mathbb{Z}^3$, then call $\vec{a} = f^{-1}\left(\vec{i}\right)$, $\vec{b} = f^{-1}\left(\vec{j}\right)$, and $\vec{c} = f^{-1}\left(\vec{k}\right)$ the inverse images of the canonical basis. The parallellopiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is called the **primitive unit cell**. Often, some other linearly independent linear combination $\vec{a'}$, $\vec{b'}$, and $\vec{c'}$ of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is chosen. The parallellopiped spanned by $\vec{a'}$, $\vec{b'}$, and $\vec{c'}$ is called a **(conventional) unit cell**.

How can we distinguish between a primitive unit cell and a conventional unit cell for a given Bravais lattice V ? A primitive unit cell is spanned by a basis of V, whereas a conventional unit cell is spanned by three vectors that are merely linearly independent.

12. INTRODUCTION TO TOPOLOGY

Topology will help us to deal with geometrical terms such as distances, angles, continuous maps, open sets, connected spaces, etc. The definition of a topological space is somewhat formal, so you may want to read it a few times and make sure you figure out what that means. Chapter 3 of the book by Stancl and Stancl [6] gives a good introduction. See also [4], [5], [8], and [7].

Definition: A **topological space** is a pair $(X, \mathbf{T})$, where X is a set and $\mathbf{T}$ a set of subsets of X with the properties (T1-3). $\mathbf{T}$ is sometimes called the **topology** of X. The elements of $\mathbf{T}$ are called **open** sets. A subset $V \subseteq X$ is called **closed**, if its complement $X \setminus V$ is open.
(T1) $X \in \mathbf{T}$, $\emptyset \in \mathbf{T}$.
(T2) If $U_1, U_2, \ldots, U_k \in \mathbf{T}$, then $\bigcap_{i=1}^{k} U_i \in \mathbf{T}$.
(T3) If $U_\lambda \in \mathbf{T}$ for $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} \in \mathbf{T}$.
What does that mean ?
(T1) The whole set and the empty set are open.
(T2) Finite intersections of open sets are also open.
(T3) *Any* unions of empty sets are also open. Λ is an index set that can be finite, denumerable, or not countable.

Let X be a topological space and x a point in X. We call a subset $V \subseteq X$ a **neighborhood** of x if $x \in V$ and there is an open set U with $U \subseteq V$. A subset $U \subseteq X$ is open if and only if U is a neighborhood for each point $x \in U$.

Example:
(E1) For any set X, let $\mathbf{T}$ be the set of all subsets of X (**discrete** topology). That means that every subset of $\mathbf{T}$ is both open and closed.
(E2) For any set X, let $\mathbf{T} = \{X, \emptyset\}$ (**indiscrete** or **blob** topology).
(E3) This is the most important example for our purposes: Let $X = \mathbb{R}^n$. For any point $x \in X$ and $\epsilon > 0$, we define the **open sphere** $K(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$, where $d(x, y)$ is the usual Euklidean distance between two points. A subset U of X is called open (i. e., $U \in \mathbf{T}$), if for each $x \in U$, there exists an $\epsilon > 0$ such that $K(x, \epsilon) \subseteq U$.
(E4) Take $X = \mathbb{R}$ (i.e., n=1 in the exercise above). It is easy to see that for $a, b \in \mathbb{R}$, an interval $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ is open and an interval $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ is closed. We see that not every set is open or closed, just look at the half-open half-closed interval $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

Definition: Maps between topological spaces X and Y can be quite boring, if they have no additional properties other than just being maps. At least, they could respect the topologies defined in X and Y. This leads to the following definition: A map $f : X \to Y$ between topological spaces X and Y is called **continuous**, if for each open set $U \subseteq Y$ its inverse image $f^{-1}(U) \subseteq X$ is also open.

Definition: A continuous map $f : X \to Y$ is called **homeomorph** (or a **homeomorphism**), if there exists a continuous map $g : Y \to X$, such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Two topological spaces are called **of the same topological type**, if they are homeomorphic.

Definition: Let X be a topological space. A set $\mathbf{B}$ of open subsets of X is called **basis of the topology**, if every open set in X can be written as a union of open sets in $\mathbf{B}$. The basis is called

countable if $\mathbf{B}$ has a countable number of elements.

Example:
(E1) The open spheres $K(x,\epsilon), x \in \mathbb{R}, \epsilon \in \mathbb{R}^+$, form a basis of the topology in $\mathbb{R}^n$. This basis is not countable.
(E2) We can, however, find a countable basis of the topology in $\mathbb{R}^n$: Just take the family of open spheres $K\left(q, \frac{1}{m}\right), q \in \mathbb{Q}^n, m \in \mathbb{N}$.

Definition: Let X be a topological space. X is called a **Hausdorff space** if for any two points $x, y \in X, x \neq y$ there are neighborhoods V of x and U of y with $U \cap Y = \emptyset$.

Example:
(E1) A discrete topological space is a Hausdorff space. In particular, the space $\mathbb{Z}$ or any finite group with the discrete topology is a Hausdorff space. We can show that a Hausdorff space with a finite number of points is discrete.
(E2) An indiscrete space containing more than one point is not a Hausdorff space.
(E3) $\mathbb{R}^n$ with the usual topology is a Hausdorff space, since for any two points $x \neq y$ we can find two open spheres $K(x,\epsilon)$ and $K(y,\epsilon)$ with empty intersection by just choosing ϵ small enough.

Definition: Let X be a topological space. A family $\mathbf{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of subsets of X is called a **cover** of X if $X = \bigcup_{\lambda\in\Lambda} U_\lambda$. The cover is called **open** (or **closed**) if all U_λ are open (or closed). If $\Gamma \subseteq \Lambda$ is a subset and the family $\{U_\lambda\}_{\lambda\in\Gamma}$ is also a cover of X we say that the cover $\{U_\lambda\}_{\lambda\in\Gamma}$ is a **subcover** of $\{U_\lambda\}_{\lambda\in\Lambda}$. $\mathbf{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is called **finite** if the set Λ is finite.

Definition: Let X be a Hausdorff topological space. X is called **compact** if any cover of X contains a finite subcover.

Example:
(E1) A discrete space is compact if and only if it is finite.
(E2) $\mathbb{R}$ is not compact, just consider the cover $\mathbf{U} = \{U_n\}_{n\in\mathbb{N}}, U_n = \{t \in \mathbb{R} \,|\, t > n\}$. If $\mathbf{U}$ had a finite subcover $\mathbf{U}' = \{U_n\}_{n\in I}$ with $I \subseteq \mathbb{N}$, then we could call n_0 the smallest element of I. For $t < n_0$, we would have $t \notin U_n$ for all $n \in I$. Therefore $\mathbf{U}'$ is not a cover of $\mathbb{R}$, which leads to a contradiction. (For this proof we had to use a special property of $\mathbb{N}$: Every subset of $\mathbb{N}$ contains a smallest element. This is not true for subsets of $\mathbb{R}$, on the other hand, just consider the open sphere $K(0,\epsilon)$.)
(E3) The unit interval $I = [0,1]$ is compact.

Theorem (Heine-Borel): Let $A \subseteq \mathbb{R}^n$ be a subset of the topological space $\mathbb{R}^n$. A is compact if and only if A is bounded and closed, where bounded means that A is contained in some open sphere $K(0,\epsilon)$ for a sufficiently large ϵ.

Exercises: (They are all very simple.)

(X1) Show that (E1) and (E2) define topological spaces.

(X2) Using (E3), find an infinite intersection of open sets that is not open.

(X3) Show that for two continuous maps $f : X \to Y$ and $g : Y \to Z$ their composition $g \circ f$ is also continuous.

(X4) Show that the identity $\mathrm{id} : X \to X, x \mapsto x$ is continuous.

Problems:

(P1) Show that a homeomorphism is bijective. Find a bijective continous map that is not a homeomorphism.

(P2) $(X, \mathbf{T})$ is a topological space. Show that X and $\emptyset$ are closed. Show that finite unions and any intersections of closed sets are also closed. (You will need de Morgan's identities: What is the complement of a union (or intersection) of sets ?)

13. CONNECTIVITY

Definition:
Let X and Y be two topological spaces. Then the **product space** $X \times Y$ is given by $X \times Y$ as sets with the following **product topology**: A subset $W \subseteq X \times Y$ is called open, if for each point $(x, y) \in W$ there are neighborhoods U of x in X and V of y in Y with $U \times V \subseteq W$.

Example:
For a topological product $X \times Y$, the **open boxes** $U \times V$ (with U open in X and V open in Y) form a basis of the topology.

Definition:
For two sets M, N with an empty intersection $M \cap N$, we can define their **sum** as $M + N = M \cup N$. If $M \cap N$ is not empty, the definition becomes more tricky. We can cause M and N to have an empty intersection by replacing M by $M \times \{1\}$ and N by $N \times \{2\}$. We therefore define the **sum of sets** as $M + N = (M \times \{1\}) \cup (N \times \{2\})$. For two topological spaces X and Y, we define their topological sum $X + Y$ (or $X \coprod Y$) as the sum of sets $X + Y$ with the following **sum topology**: A subset $W \subseteq X + Y$ is called open, if $W = U + V$ with $U \subseteq X$ and $V \subseteq Y$ open.

Example:
If X is a topological space, then we have the **trivial sum** $X = X + \emptyset = \emptyset + X$.

Definition:
A topological space X is **connected**, if it can only be written as a trivial sum of two topological spaces, that is $X = Y + Z$ implies either $Y = \emptyset$ or $Z = \emptyset$. In other words, a topological space is connected, if only the empty set and the whole space are open *and* closed at the same time.

Example:
The unit interval $I = [0, 1] \subseteq \mathbb{R}$ is connected.

14. ALGEBRAIC TOPOLOGY AND THE FUNDAMENTAL GROUP

Let $I \subseteq \mathbb{R}$ be the interval $I = [0, 1]$. A **path** or **arc** in a topological space X is a continuous map $f : I \to X, t \mapsto f(t)$. You might think of X as being the three-dimensional space $\mathbb{R}^3$, t being time, and the path describing the motion of a particle in space. The **end points** of the path are called the **initial** and **terminal points** of the path. For a point $x \in X$, the **trivial path** ϵ_x is defined by $\epsilon_x(t) = x$ for all $t \in I$ (particle at rest).

A topological space X is called **arcwise connected** or **pathwise connected**, if any two points $x, y \in X$ can be connected (joined) by a path. An arcwise connected topological space X is also connected, but the converse statement is not necessarily true. The **arc components** of X are the maximal arcwise-connected subsets of X.

A space X is called **locally arcwise connected** if for any point $x \in X$ each neighborhood of x contains an arcwise connected neighborhood of x.

Definition: Let $f_0, f_1 : I \to X$ be two paths in a topological space X such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$, i.e., the two paths have the same endpoints. We say that these two paths are **equivalent**, denoted by $f_0 \sim f_1$, if there exists a continous map $f : I \times I \to X, (t,s) \mapsto f(t,s)$ such that for $s, t \in I$ we have $f(t,0) = f_0(t)$, $f(t,1) = f_1(t)$ and $f(0,s) = f_0(0) = f_1(0)$, $f(1,s) = f_0(1) = f_1(1)$. Intuitively, you can think of t as time and s some parameter. The two paths f_0 and f_1 are called equivalent, if one can be continously deformed into the other in the space X. You may show that this defines an **equivalence relation**. We call $[f]$ the **equivalence class** of a path f, but immediately drop the brackets in order to keep our notation simple.

A path, or a path class, $f : I \to X$ with endpoints $x = f(0)$ and $y = f(1)$ is called **closed**, or a **loop**, if $x = y$, that is the initial and terminal points are the same.

Definition:
For two paths $f, g : I \to X$ with $f(1) = g(0)$ (i.e., the terminal point of path f and the initial point of path g are the same) we can define their **product** fg by letting $(fg)(t) = f(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(fg)(t) = g(2t-1)$ for $\frac{1}{2} \leq x \leq 1$.

In the same way, we can also define the product of two equivalence classes of paths. We can show that this product of equivalence classes is well-defined (that is independent of the representatives of the equivalence classes) and associative.

Theorem and Definition:
Let X be a topological space and $x \in X$. The set of equivalence classes of loops based at $x = f(0) = f(1)$ forms a group with the product defined above. The neutral element is given by the trivial path ϵ_x, the inverse element is given by $f^{-1} : I \to X, t \mapsto f^{-1}(t) = f(1-t)$, i.e., the inverse path is obtained by traversing the path f in the opposite direction. This group is called the **fundamental group** or **Poincare group** of X at the base point x and denoted by $\pi(X, x)$.

Theorem:
If X is arcwise connected and $x, y \in X$, then the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic. That means that the structure and properties of the fundamental group are independent of the base point and therefore only a property of the arcwise connected topological space X.

Definition:
A topological space X is called **simply connected**, if it is arcwise connected and the fundamental group $\pi(X, x) = \{1\}$ for some (and hence any) point $x \in X$.

If we have a continuous map $\phi : X \to Y$ and paths $f_0, f_1 : I \to X$ in X, then ϕ induces paths $\phi(f_0)$ and $\phi(f_1)$ in Y. If f_0 and f_1 are equivalent, so are $\phi(f_0)$ and $\phi(f_1)$. We can therefore say that ϕ induces a map $\phi_* : \pi(X, x) \to \pi(Y, \phi(x))$ between the fundamental groups of X and Y. We can show that this map is a homomorphism of groups. i.e., for equivalence classes of paths α, β in X, we have: (i) $\phi_*(\alpha\beta) = \phi_*(\alpha)\phi_*(\beta)$. (ii) For any point $x \in X$, $\phi_*(\epsilon_x) = \epsilon_{\phi(x)}$. (iii)

$\phi_*\left(\alpha^{-1}\right) = \left(\phi_*\left(\alpha\right)\right)^{-1}$. Furthermore, if ϕ is a homeomorphism, then ϕ_* is an isomorphism of groups.

Definition:
Two continuous maps $\phi_0, \phi_1 : X \to Y$ are **homotopic** if and only if there exists a continous map $\phi : X \times I \to Y$ such that, for $x \in X$, $\phi\left(x,0\right) = \phi_0\left(x\right)$ and $\phi\left(x,1\right) = \phi_1\left(x\right)$. If two maps ϕ_0 and ϕ_1 are homotopic, we will denote this by $\phi_0 \simeq \phi_1$. This defines an equivalence relation on the set of all continuous maps $X \to Y$. The equivalence classes are called **homotopy classes** of maps. (The geometrical content of this definition is the following: Two continuous maps are homotopic if one can be continuously deformed into the other. This continuous deformation ϕ is often called the **homotopy**.)

Definition:
A subset A of a topological space X os called a **retract** of X if there exists a continuous map $r : X \to A$ (called a **retraction**) such that $r\left(a\right) = a$ for any $a \in A$.

Example:
Let X be the Möbius strip. A retract of X is then the center circle of the strip.

Definition:
A subset A of a topological space X is called a **deformation retract** if there exists a retraction r homotopic to the identity map $\mathrm{id} : X \to X$, or in other words if there is a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that $f\left(x,0\right) = x$, $f\left(x,1\right) = r\left(x\right)$ for all $x \in X$ and $f\left(a,t\right) = a$ for all $a \in A, t \in I$.

15. COVERING SPACES

In this chapter (and from now on), we assume that all topological spaces are arcwise connected and locally arcwise connected and simply write *space*.

Definition:
Let X be an arcwise and locally arcwise connected topological space. A **covering space** of X is a pair $\left(\tilde{X}, p\right)$ consisting of a space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ such that the following condition holds: Each point $x \in X$ has an arcwise connected open neighborhood U such that each arc component of $p^{-1}\left(U\right)$ is mapped topologically onto U by p (in particular, it is assumed that $p^{-1}\left(U\right)$ is not empty). Any open neighborhood U that satisfies the condition just stated is called an **elementary neighborhood**. The map p is often called projection.

Example:
(E1) This is one of the simplest and most important examples for a covering space. Consider the topological spaces $X = S^1 = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$ (that is the unit circle in the plane) and $\tilde{X} = \mathbb{R}$. If we define the projection $p : \mathbb{R} \to S^1, t \mapsto p\left(t\right) = \left(\sin t, \cos t\right)$, then the pair $\left(\mathbb{R}, p\right)$ is a covering space of the unit circle S^1. Any open subinterval of the circle can serve as an elementary neighborhood.
(E2) A trivial example is $\tilde{X} = X$ with p being the identity of X.
(E3) Use polar coordinates $\left(r, \theta\right)$ to denote the points in the plane $\mathbb{R}^2$. Then the unit circle S^1 is defined by the condition $r = 1$. For any integer $n \in \mathbb{Z}$, positive or negative, define a map $p_n : S^1 \to S^1$ by the equation $p_n\left(1, \theta\right) = \left(1, n\theta\right)$. The map p_n wraps the circle around itself n times.

It is readily seen that for $n \neq 0$, the pair (S^1, p_n) is a covering space of S^1. Once again, any proper open interval in S^1 is an elementary neighborhood.

16. MANIFOLDS AND LIE GROUPS

Much of the discussion about Lie groups and their representations was taken from the book by Bröcker and tom Dieck [10].

A differentiable manifold M^n is a topological space that is locally homeomorphic (locally looks like) a subset of $\mathbb{R}^n$. We will usually deal with manifolds that are subsets of $\mathbb{R}^m$ (or $\mathbb{C}^{m/2} \cong \mathbb{R}^m$ for m even) with some $m > n$, therefore we don't have to worry about such things as Hausdorff or countable topological basis. Physicists normally use the word **smooth surface** rather than manifold. Here is the complete definition (following Bröcker and tom Dieck [10]):

Definition: An n-**dimensional (differentiable) manifold** M^n is a Hausdorff topological space with a countable (topological) basis, together with a maximal **differentiable atlas**. This atlas consists of a family of **charts** $h_\lambda : U_\lambda \to U'_\lambda \subseteq \mathbb{R}^n$, where the **domains of the charts**, $\{U_\lambda\}$, form an open cover of M^n, the U'_λ are open in $\mathbb{R}^n$, the charts (or **local coordinates**) h_λ are homeomorphisms, and every **change of coordinates** $h_{\lambda\mu} = h_\mu \circ h_\lambda^{-1}$ is differentiable on its domain of definition $h_\lambda(U_\lambda \cap U_\mu)$. Here the word **differentiable** means infinitely often differentiable. We will use **smooth**, differentiable, and C^∞ synonymous. The atlas is **maximal** in the sense that it cannot be enlarged to another differentiable atlas by adding more charts, so any chart which could be added to the atlas in a consistent fashion is already in the atlas.

Definition: Let M^m and N^n be differentiable manifolds. A continuous map $f : M \to N$ is called **differentiable** if, after locally composing with the charts of M and N, it induces a differentiable map of open subsets of Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$.

Definition: A **Lie group** G is a differentiable manifold G which is also a group such that the group multiplication $\mu : G \times G \to G$ and the map $\iota : G \to G, g \mapsto g^{-1}$ are differentiable maps. A **homomorphism of Lie groups** is a homomorphism of groups which is also a differentiable map of manifolds.

Example:
(E1) Every finite-dimensional vector space with its additive group structure is a Lie group in a canonical way. Thus, up to isomorphism, we have the groups $\mathbb{R}^n$, $n \in \mathbb{N}_0$.
(E2) Any denumerable or finite group is a Lie group, since each of the group elements is locally homeomorphic to the null space $\mathbb{R}^0 = \{0\}$. Therefore, the representations of finite or denumerable groups are only a special case of the representations of Lie groups.
(E4) The **torus** $T^n = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n = (S^1)^n$ is a Lie group. Here, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle viewed as a multiplicative subgroup of $\mathbb{C}$, and the isomorphism $\mathbb{R}/\mathbb{Z} \to S^1$ is induced by $t \mapsto \exp(2\pi i t)$. The n-fold product of the circle with itself has the structure of an Abelian Lie group with the following example:
(E3) If G and H are Lie groups, so is their product $G \times H$ with the direct product of the group and manifold structures on G and H. It will turn out that every connected Abelian Lie group is isomorphic to the product of a vector space and a torus.
(E5) The **real projective space** $\mathbb{R}P^n$ of lines through the origin in $\mathbb{R}^{n+1}$ may be given the structure of an n-dimensional manifold. The elements of $\mathbb{R}P^n$ are equivalence classes in $\mathbb{R}^{n+1}$, where two points x and x' in $\mathbb{R}^{n+1}$ are called equivalent if there is an $r \in \mathbb{R}$ such that $x = rx'$. The elements of $\mathbb{R}P^n$ are isomorphic to the equivalence classes of all quantum mechanical **wave functions** in an

n-dimensional Hilbert-space that are normalized to some expectation value (usually 1), but wave functions with different complex phase vectors are considered to be different entities.
(E5') In the same way, we can define the **complex projective space** $\mathbb{C}P^n$ as lines through the origin in $\mathbb{C}^{n+1}$, that is the space of equivalence classes $[z]$, where $z, z' \in \mathbb{C}^{n+1}$ are called equivalent if there is an $s \in \mathbb{C}$ with $z = sz'$. $\mathbb{C}P^n$ is also a manifold (but not a Lie group, because there is no multiplication of vectors in $\mathbb{C}^{n+1}$). The elements of $\mathbb{C}P^n$ are isomorphic to the equivalence classes of all quantum mechanical **wave functions** that are equal up to a complex phase.
(E6) Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. The set $\mathrm{Aut}\,(V)$ of linear automorphisms of V is an open subset of the finite-dimensional vector space $\mathrm{End}\,(V)$ of linear maps $V \to V$, because $\mathrm{Aut}\,(V) = \{A \in \mathrm{End}\,(V) \mid \det(A) \neq 0\}$ and the determinant is a continuous function. Thus $\mathrm{Aut}\,(V)$ has the structure of a differentiable manifold. After the introduction of coordinates in V (using a basis of V), the group operation of $\mathrm{Aut}\,(V)$ is matrix multiplication, which is algebraic and hence differentiable. Therefore, $\mathrm{Aut}\,(V)$ has a canonical structure as a Lie group, and we get the groups (**general linear groups**)

$$\mathrm{GL}\,(n, \mathbb{R}) = \mathrm{Aut}_{\mathbb{R}}\,(\mathbb{R}^n) \quad \text{and} \quad \mathrm{GL}\,(n, \mathbb{C}) = \mathrm{Aut}_{\mathbb{C}}\,(\mathbb{C}^n)\,.$$

Linear maps $\mathbb{R}^n \to \mathbb{R}^k$ may be described by $(k \times n)$-matrices (using the canonical bases), and, in particular, $\mathrm{GL}\,(n, \mathbb{R})$ is canonically isomorphic to the group of invertible $(n \times n)$-matrices. Thus we will think of $\mathrm{GL}\,(n, \mathbb{R})$ and $\mathrm{GL}\,(n, \mathbb{C})$, and their classical subgroups $\mathrm{SL}\,(n, \mathbb{R})$, $\mathrm{SL}\,(n, \mathbb{C})$, $\mathrm{O}\,(n)$, $\mathrm{SO}\,(n)$, $\mathrm{U}\,(n)$, $\mathrm{SU}\,(n)$ as matrix groups.
(E7) The group $\mathrm{GL}\,(n, \mathbb{R})$ has two connected components (as a topological space) on which the sign of the determinant is constant. Automorphisms with positive determinant form an open and closed subgroup $\mathrm{GL}^+\,(n, \mathbb{R})$. It is connected, because $\{+1\}$ is connected in $\mathbb{R}$ and the determinant is a continuous mapping.
(E8) A closed subgroup of a Lie group is a Lie group. ("Closed" is used in its topological sense here.)
(E9) The quotient of a Lie group by a closed normal subgroup is a Lie group.
(E10) As a result of (E8), we get the **special linear groups** over $\mathbb{R}$ and $\mathbb{C}$,
$\mathrm{SL}\,(n, \mathbb{R}) = \{A \in \mathrm{GL}\,(n, \mathbb{R}) \mid \det(A) = 1\}$ and $\mathrm{SL}\,(n, \mathbb{C}) = \{A \in \mathrm{GL}\,(n, \mathbb{C}) \mid \det(A) = 1\}$.
(E11) As a result of (E9), we get the **projective groups** $\mathrm{PGL}\,(n, \mathbb{R}) = \mathrm{GL}\,(n, \mathbb{R})\,/\mathbb{R}^*$ and $\mathrm{PGL}\,(n, \mathbb{C}) = \mathrm{GL}\,(n, \mathbb{C})\,/\mathbb{C}^*$, where $\mathbb{R}^*$ and $\mathbb{C}^*$ are embedded as the subgroups of scalar multiples of the identity matrix. The projective groups are groups of transformations of projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ and important for quantum mechanics.
(E12) The **orthogonal groups** $\mathrm{O}\,(n) = \{A \in \mathrm{GL}\,(n, \mathbb{R}) \mid A^t \cdot A = E\}$, where A^t denotes the transpose matrix and E the identity matrix. The elements of $\mathrm{O}\,(n)$ are called **orthogonal** matrices. Analogously there is the **unitary group** $\mathrm{U}\,(n) = \left\{A \in \mathrm{GL}\,(n, \mathbb{C}) \mid A^\dagger A = E\right\}$, where $A^\dagger$ is the conjugate transpose of A. The elements of $\mathrm{U}\,(n)$ are called **unitary matrices**. If we define the standard **Euklidean scalar product** on $\mathbb{R}^n$ and the standard **Hermitian product** on $\mathbb{C}^n$, then $\mathrm{O}\,(n)$ and $\mathrm{U}\,(n)$ consist of those automorphisms (matrices) that preserve these scalar products (or **inner products**).
(E13) $\mathrm{O}\,(n)$ is also split into two connected components by the values of ± 1 of the determinant, and one of these is the **special orthogonal group** $\mathrm{SO}\,(n) = \{A \in \mathrm{O}\,(n) \mid \det(A) = 1\}$. The unitary group $\mathrm{U}\,(n)$ is connected, and has the subset $\mathrm{SU}\,(n) = \{A \in \mathrm{U}\,(n) \mid \det(A) = 1\}$ called the **special unitary group**. All these groups ($\mathrm{O}\,(n)$, $\mathrm{U}\,(n)$, $\mathrm{SO}\,(n)$, and $\mathrm{SU}\,(n)$) are compact, since they are closed and bounded in the finite-dimensional vector space $\mathrm{End}\,(V)$ (Heine-Borel).

17. TANGENT SPACES AND LIE ALGEBRAS

I do not want to talk too much about Lie algebras here, since they really belong into high-energy theory. There are many interesting things to note about Lie algebras. For example, there is a classification of Lie algebras that allow only a denumerable (but still infinite) number of theories for the universe. Let me briefly note a few things which are important for classical and quantum mechanics.

Let B be vector space over the field K. (You can also define an algebra over a commutative ring, but we will skip this here.) An **(associative) algebra** is a vector space B together with a K-bilinear map $* : B \times B \to B$ that defines a multiplication within B that makes $(B, *)$ a ring. The bilinearity implies that $(ax)(by) = (ab)(xy)$ for all $a, b \in K$ and $x, y \in B$.

Let B be an algebra. We write the multiplication as $[] : B \times B \to B, (x, y) \mapsto [x, y]$. B is called a **Lie algebra**, if $[x, x] = 0$ (L1) and $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in B$ (L2). (This equation is called **Jacobi's identity**.) We note that $[x, x] = 0$ implies $[x, y] = -[y, x]$ for all $x, y \in B$ (L3).

Example:
(E1) Let V be an n-dimensional K-vector space and $B = \mathrm{End}(V)$ be the ring of $n \times n$-matrices over K. The product $[x, y] = xy - yz$ for $x, y \in B$ makes B a Lie algebra.
(E2) Let B be the ring of C^∞-differentiable functions $f(\vec{r}, \vec{p}, t)$ (with coordinates $\vec{r}$ and momenta $\vec{p}$ being the parameters) from K^{2n} into K (with K being $\mathbb{R}$ or $\mathbb{C}$), with $[f, g]$ defined by the Poisson brackets of classical mechanics:

$$[f, g] \;=\; \mathrm{grad}_r f \cdot \mathrm{grad}_p g - \mathrm{grad}_p f \cdot \mathrm{grad}_r g \;=\; \frac{df}{d\vec{r}} \cdot \frac{dg}{d\vec{p}} - \frac{df}{d\vec{p}} \cdot \frac{dg}{d\vec{r}}. \tag{17.1}$$

This makes B into a Lie algebra. See Goldstein, Classical Mechanics, chapter 8 [12] for a proof of Jacobi's identity.
(E3) Trivial (but important) example: Let $[X, Y] = 0$ for all $X, Y \in B$.

Let us now talk about the relationship between Lie algebras and Lie groups, without getting too deep into mathematical terms. This relationship is very geometrical in nature and has to do with an intuitive statement from calculus: *Locally, everything is linear.* In order to explain this statement, we have to define what *locally* is supposed to mean. This will lead to the definition of the term *germ*.

Definition: Let G be a Lie group. Since G is also an n-dimensional differentiable manifold (that is a smooth surface), we understand what the term **tangent space** is supposed to mean. In mathematical terms, the tangent space at a point p of a submanifold $M \subseteq \mathbb{R}^n$ is the space of all **velocity vectors** $\dot{\alpha}(0)$ of arcs $\alpha : \mathbb{R} \to M$ with $\alpha(0) = p$. Let us now define the term tangent space for the general case of manifolds.

First, in less mathematical terms: Obviously, the tangent space for an n-dimensional manifold T_pM is supposed to be an n-dimensional vector space and therefore isomorphic to $\mathbb{R}^n$. Therefore, for a given vector v in $\mathbb{R}^n$, the corresponding tangent vector X is the directional derivative in the direction of v. Now the accurate definition:

Definition: Let M be an n-dimensional manifold with $p \in M$. Let N be another manifold and $f : M \to N$ and $g : M \to N$ differentiable maps from M into N. For a point $p \in M$ with $f(p) = q \in N$, these two maps are said to have **equal germs at** p, if for some neighborhood U of p their restrictions to U are identical, that is $f|U = g|U$. Having equal germs is an equivalence relation. An equivalence class is called a **germ** and denoted $f : (M, p) \to (N, q)$. Such a germ is represented by a map $f : U \to N$, where U is a neighborhood of p, and $g : V \to N$ represents the same germ if f and g agree on a smaller neighborhood $W \subseteq U \cap V$. The set $\mathcal{E}_p$ of all germs of real-valued functions $\phi : (M, p) \to \mathbb{R}$ is an $\mathbb{R}$-algebra in a natural way, addition and multiplication being done on representatives.

Definition: Let M be an n-dimensional manifold with $p \in M$. A **tangent vector** at p is a linear map $X : \mathcal{E}_p \to \mathbb{R}$ satisfying the following product rule (a **derivation** of the $\mathbb{R}$-algebra $\mathcal{E}_p$):

$$X(\phi \cdot \psi) \;=\; X(\phi) \cdot \psi(p) \;+\; \phi(p) \cdot X(\psi). \tag{17.2}$$

One should think of $X(\phi)$ as the directional derivative of ϕ in the direction X. The set T_pM of all tangent vectors at p is a real vector space in a natural way and is called the **tangent space** of M at the point p.

Definition: Now let M and N be two manifolds and $f : M \to N$ be a differentiable map. The germ of this map at a point p, that is $f : (M, p) \to (N, q)$ induces a homomorphism of $\mathbb{R}$-algebras

$$f^* : \mathcal{E}_q \to \mathcal{E}_p, \quad \phi \mapsto \phi \circ f \tag{17.3}$$

and hence the **tangent map** (also called the **differential**)

$$T_pf : T_pM \to T_qN, \quad X \mapsto X \circ f^*. \tag{17.4}$$

That is to say that a map $f : M \to N$ induces a tangent map T_pf between the tangent spaces T_pM and T_qN with

$$T_pf(X)\,\phi \;=\; X(\phi \circ f). \tag{17.5}$$

The map $f \mapsto T_pf$ is **functorial**, which means that $T_pf(\mathrm{id}) = \mathrm{id}$ and that it is compatible with compositions.

Definition: Let us now apply this tangent vector formalism about manifolds to the special case of Lie algebras. Let G be a Lie group with unit element e. The vector space $LG := T_eG$ is called the **Lie algebra** of G. (The term "algebra" is not yet justified, but we will explain the algebra structure soon.) A homomorphism of Lie groups $f : G \to H$ induces a homomorphism of Lie algebras $Lf : LG \to LH$ in a functorial fashion.

Let us now define, what $[X, Y]$ is supposed to be for a Lie group G with its Lie algebra LG and $X, Y \in LG$. X and Y are tangent vectors, that is derivations or directional derivatives. Their product $[X, Y]$ also has to be a tangent vector. X as a tangent vector acts on the germ ϕ of a function at the point $g \in G$. The resulting directional derivative $X\phi$ may also be viewed as the germ of a function, therefore it makes sense to define the **Lie product**

$$[X,Y]\,\phi \;=\; X\,(Y\phi) - Y\,(X\phi)\,. \tag{17.6}$$

An easy calculation shows that $[X,Y]$ satisfies the product rule and therefore is another tangent vector. The properties of the Lie product are such that the Lie algebra conditions (L1) and (L2) are met, therefore the name Lie algebra is well deserved for this construction.

Definition: A **one-parameter group** of a Lie group G is a homomorphism of Lie groups $\alpha : \mathbb{R} \to G, t \mapsto \alpha\,(t)$ (the homomorphism, not just its image !). The correspondence $\alpha \mapsto \dot{\alpha}\,(0) \in LG$ defines a canonical bijection between the set of one-parameter groups of G and the Lie algebra of G.

In fact, for a given one-parameter group α, the corresponding tangent vector (directional derivative) X^{α} acts on a germ represented by $\phi : G \to \mathbb{R}$ as follows:

$$\left(X^{\alpha}\phi\right)(g) \;=\; \left.\frac{\partial}{\partial t}\right|_{t=0} \phi\,(g\alpha\,(t))\,. \tag{17.7}$$

If $\beta\,(s)$ is another one-parameter group, then the second derivative is given by

$$\left(X^{\beta}X^{\alpha}\phi\right)(g) \;=\; \left.\frac{\partial^2}{\partial s\partial t}\right|_{\substack{s=0\\t=0}} \phi\,(g\alpha\,(t)\,\beta\,(s))\,. \tag{17.8}$$

If we have a tangent vector $X \in LG$, then the corresponding one-parameter group $\alpha\,(t)$ is given by the integral curve that solves the differential equation $\dot{\alpha}\,(t) = X\,(\alpha\,(t))$.

Example:
Let us now calculate the Lie algebras and Lie products for some of the most common Lie groups:
(E1) A finite-dimensional vector space V (such as K^n), interpreted as a Lie group with unit element 0, coincides with LV, and the derivative of the germ of the real-valued function $\phi : V \mapsto \mathbb{R}, t \mapsto \phi\,(t)$ corresponding to $v \in V$ is given by

$$X^{v}\phi \;=\; \left.\frac{\partial}{\partial t}\right|_{t=0} \phi\,(tv) \;=\; v \cdot \text{grad}\;\phi\,(0)\,. \tag{17.9}$$

For a vector $v \in V$, the corresponding one-parameter group is

$$\alpha^{v} : t \mapsto \alpha^{v}\,(t) = tv. \tag{17.10}$$

(E2) Similarly, the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ has $\mathbb{R}^n$ as its Lie algebra. This is not surprising, since T^n locally looks like $\mathbb{R}^n$. For an element $v \in T^n$, the corresponding one-parameter group is

$$\alpha^{v} : t \mapsto \alpha_{v}\,(t) = tv \bmod \mathbb{Z}^n. \tag{17.11}$$

(E3) The Lie product of a Lie algebra derived from an Abelian group G (such as a vector space V or the torus T^n) is trivial, that is $[X,Y] = 0$ for $X, Y \in LG$. This is obvious from Eq. (17.8).
(E4) For a vector space V, the group of linear automorphisms $G = \text{Aut}\,(V)$ has its Lie algebra $LG =$

End (V), the vector space of all linear endomorphisms of V, since Aut (V) is an open submanifold of End (V). The one-parameter group given by $A \in \text{End}(V)$ is

$$\alpha^A : \mathbb{R} \to \text{Aut}(V), \quad t \mapsto \exp(tA) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (tA)^\nu . \tag{17.12}$$

Indeed, we see that $\dot{\alpha}^A(0) = A$, but is α^A a homomorphism ? Yes, it is:

$$\alpha^A(s+t) = \exp(sA + tA) = \exp(sA)\exp(tA) = \alpha^A(s)\,\alpha^A(t), \tag{17.13}$$

since sA and tA commute. **Physicists' convention:** For a complex vector space, physicists replace the one-parameter group given above by

$$\alpha^A : \mathbb{R} \to \text{Aut}(V), \quad t \mapsto \exp(itA) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (itA)^\nu . \tag{17.14}$$

If we calculate the second derivative in Eq. (17.8) explicitly, we find that for $A, B \in \text{End}(V)$ the Lie product is given by $[A, B] = AB - BA$. This example will be very important for us, since we find it in many applications from classical mechanics, quantum mechanics, and solid-state physics.
(E5) For the Lie group $G = \text{SO}(n) \subseteq \text{Aut}(V)$ we compute its Lie algebra $\text{so}(n) = L\text{SO}(n) \subseteq \text{End}(\mathbb{R}^n)$ as follows: For $X \in \text{so}(n)$ we have the one-parameter group α^X with $\alpha^X(t) \in \text{SO}(n)$. We remember that for a special orthogonal matrix A its inverse is given by the transpose matrix A^t and therefore calculate modulo t^2:

$$E = \left[\alpha^X(t)\right]^t \alpha^X(t) = (E + tX)^t (E + tX) = E + t\left(X^t + X\right) \tag{17.15}$$

Therefore $X^t + X = 0$, and we find that $\text{so}(n)$ consists of the skew-symmetric matrices. We can also show that all skew-symmetric matrices X indeed generate one-parameter groups in $\text{SO}(n)$. From this we conclude that the dimension of $\text{so}(n)$ is $\frac{1}{2}n(n-1)$. Similar calculations allow us to determine the Lie algebras of other linear groups.
(E6) The Lie algebra $\text{u}(n) = L\text{U}(n) \subseteq \text{End}(\mathbb{R}^n)$ is given by all skew-Hermitian matrices (for the mathematicians). For us physicists (with the extra factor i in the exponential), the Lie algebra $\text{u}(n)$ consists of all Hermitian matrices, as we are used to from quantum mechanics. This shows that $\dim \text{U}(n) = n^2$.
(E7) For the group of special linear matrices $\text{SL}(n, \mathbb{R})$ with determinant 1, its Lie algebra consists of all matrices with zero trace.
(E8) $\text{su}(n) = L\text{SU}(n)$ is the Lie algebra of all Hermitian matrices with trace zero (physicists convention).

Definition: Given a tangent vector $X \in LG$ at the unit e of a Lie group G, there is a one-parameter group $\alpha^X : \mathbb{R} \to G$ with $\dot{\alpha}^X(0) = X$. The map

$$\exp : LG \to G, \quad X \mapsto \alpha^X(1) \tag{17.16}$$

is called the **exponential map**. This map is differentiable, and its differential at the origin is the identity, that is

$$\left.\frac{\partial}{\partial t}\right|_0 \exp(tX) = X \quad \text{and therefore} \quad T_0 \exp = \mathrm{id}_{LG}. \tag{17.17}$$

We have already defined the exponential map for the linear subgroups of $\mathrm{Aut}(V)$.

Theorem: The exponential map is **natural**, that is, it is compatible with compositions. It also is a local diffeomorphism, because it is the identity at the origin and therefore invertible. This implies that a homomorphism of connected Lie groups is determined by its differential at the unit element.

Exercises:

(X1) Show that the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ is given by all real matrices with zero trace. What is the dimension of this Lie group (and its Lie algebra) ?

(X2) Show that the property (L3) follows from (L1).

18. ELEMENTARY REPRESENTATION THEORY

Representations of compact Lie groups are the chief mathematical objects of interest in this course. Groups are intended to describe symmetries of geometric and other mathematical objects, such as crystals. Representations are symmetries of some of the most basic objects in geometry and algebra, namely vector spaces.

As we have seen above, finite groups are a special case of Lie groups. Therefore, it is sufficient to treat representations of Lie groups. This will not complicate things at all. If you feel that you have taken the wrong class (the solid-state group theory class and not the high-energy group theory class), then simply disregard the word "Lie" for the rest of this chapter. In order to emphasize this, I will sometimes put "Lie" in parentheses.

We will state (and maybe prove) some fundamental theorems about representations here. The tool used in these proofs will be integration. For a finite group G, the symbol $\int_G$ is supposed to mean the summation over all group elements and then dividing the sum by the order of the group. For a Lie group, the definition of the integral is quite involved, but that does not have to concern us here. The integral is always normalized, therefore we have to divide by the volume of the Lie group when carrying out the calculations. For a finite group, the integral therefore means summing over all elements in the group and dividing by the number of elements (order of the group).

Now you should understand why we will only deal with compact (or finite) groups. An integral over a non-compact space (such as $\mathbb{R}$ or an open subset thereof) or an infinite series (in the case of $G = \mathbb{Z}$) may be divergent and not exist. Therefore, the theory of representations of non-compact groups is much more difficult. Some of the more fundamental theorems for compact groups may not be valid. The most important example for a non-compact group is $\mathbb{Z}$, the Abelian (additive) group of integral numbers. This group is quite important for solid-state physics, since it is the basis for translational symmetry. In order to avoid such mathematical problems, we introduce periodic boundary conditions, reducing the non-compact group $\mathbb{Z}$ to a finite group $\mathbb{Z}/n\mathbb{Z}$. Then, the theorems in this paragraph will also apply to the translation group, but keep in mind that everything depends on the periodic boundary conditions.

We begin by considering finite-dimensional vector spaces over the complex numbers $\mathbb{C}$. Later we will indicate the modifications necessary for working with real vector spaces and infinite-dimensional vector spaces (such as the Hilbert spaces of quantum-mechanical wave functions).

Definition: A **representation** of the (Lie) group G on the (finite-dimensional complex) vector space V is a (continuous) action $\rho : G \times V \to V$ of G on V such that for each $g \in G$ the translation $l_g : v \mapsto \rho(g, v)$ is a linear map. We call the pair (V, ρ) a **complex representation** and V the **representation space**. Sometimes, we will abuse notation and call V a **complex G-module**. The dimension of V (as a complex vector space) is called the **dimension** $\dim V = |V|$ of the representation. We usually denote $\rho(g, v)$ by gv so saying that ρ is an action means that

$$ev = v \quad \text{and} \quad (gh)\, v = g\,(hv)\,. \tag{18.1}$$

Written in terms of translations, these equations become $l_e = \mathrm{id}_V$ and $l_g \circ l_h = l_{gh}$.

The definition given above is called the **geometric form** of a representation. For us physicists, the **numerical form** of the representation may be more convenient. From the equations stated above, we conclude that l_g is a linear automorphism of V with inverse $l_{g^{-1}}$ and the map

$$l : G \to \mathrm{Aut}_{\mathbb{C}}(V)\,, \quad g \mapsto l_g \tag{18.2}$$

is a homomorphism. A **matrix representation** of G is a continuous homomorphism $l : G \to \mathrm{GL}(n, \mathbb{C})$.

Definition: A representation is called **faithful** (or **true**) if the associated homomorphism $G \to \mathrm{Aut}(V)$ is injective. (This means that no two group elements correspond to the same matrix.) Let us state briefly without proof that every compact Lie group has a faithful representation and is therefore isomorphic to a closed subgroup of a matrix group.

Example:
Consider the equilateral triangle group and $V = \mathbb{C}^2$. Then $\mathrm{Aut}(V) = GL(2, \mathbb{C})$ and the map associating a group element $A, ..., F$ with one the 2×2 matrices is a faithful representation.

Example:
(E1) The representations of $\mathrm{SU}(n)$, $\mathrm{U}(n)$, and $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ in which elements of the stated Lie groups simply operate by matrix multiplication are called the **standard representations**.
(E2) A representation is called **trivial** if each group element acts as the identity.

Definition: Let G be a Lie group and V and W be complex G-modules. A **morphism** is a linear map $f : V \to W$ which is equivariant (that is it is compatible with the operation of G defined on V and W), i.e., which satisfies $f(gv) = gf(v)$ for all $g \in G$ and $v \in V$. Morphisms are also called **intertwining operators**. They form a category, and as usual an isomorphism is a morphism which has an inverse. Isomorphic representations are also called **equivalent** representations.

Let us make this definition a little more obvious for the case $V = W$. V is an n-dimensional complex vector space with a basis. The operation of the group on V is defined by a map

$$\alpha : G \to \mathrm{Aut}(V)\,, \quad g \mapsto \alpha(g) \tag{18.3}$$

Given a basis, the map $f : V \to V$ may be described by an $n \times n$ matrix. We may also assume that the $\alpha(g)$ are $n \times n$ matrices. With this notation, f is a morphism if and only if

$$M\alpha(g) = \alpha(g)M. \tag{18.4}$$

In other words, a morphism is described by a matrix M which commutes with the matrices $\alpha(g)$ for all $g \in G$.

Definition: Let α and β be two matrix representations $G \to \mathrm{GL}(n, \mathbb{C})$ and $V_\alpha = (\mathbb{C}^n, \rho_\alpha)$, $V_\beta = (\mathbb{C}^n, \rho_\beta)$ be the corresponding representations on $\mathbb{C}^n$. Then using the correspondence between linear maps $f : V_\alpha \to V_\beta$ and complex $(n \times n)$-matrices, we see that V_α and V_β are isomorphic if and only if there is an invertible matrix M such that

$$M\alpha(g)M^{-1} = \beta(g) \quad \text{for all} \quad g \in G. \tag{18.5}$$

If two representations α and β are related as in (18.5), they are said to be **similiar** or **conjugate**. This should not be confused with complex conjugation.

Definition: If V is a complex G-module, an (Hermitian) inner product $V \times V \to \mathbb{C}, (u, v) \mapsto \langle u, v\rangle$ is called **G-invariant** if $\langle gu, gv\rangle = \langle u, v\rangle$ for all $g \in G$ and $u, v \in V$. A representation together with a G-invariant inner product is called a **unitary representation**. If we choose an orthonormal basis for the space V of a unitary representation, then the associated matrix representation is a homomorphism $G \to \mathrm{U}(n)$. In other words, the matrices corresponding to the group elements are unitary matrices.

Theorem: Let V be a representation of the compact group G. Then V possesses a G-invariant inner product. To be specific, if $b : V \times V \to \mathbb{C}$ is any inner product, then we can construct a G-invariant inner product

$$c(u, v) = \int_G b(gu, gv)\, dg \tag{18.6}$$

In other words, for a compact Lie group, given any representation, we can find a unitary representation. We can therefore assume for the future that all representations are unitary (if the group is compact).

Corollary: Since every representation has a G-invariant inner product, it may be assumed to be unitary. Therefore, the matrix representation for any compact Lie group may be assumed to be a homomorphism $G \to \mathrm{U}(n)$.

Definition: Let V be a G-module. A subspace $U \subseteq V$ which is G-invariant (that is $gu \in U$ for all $g \in G$ and $u \in U$) is called a **submodule** of V or a **subrepresentation**. A nonzero representation V is called **irreducible** if it has no submodules other than $\{0\}$ and V. A representation which is not irreducible is called **reducible**.

Proposition: Let G be a compact group. If U is a G-submodule of the G-module V, then there is a complementary submodule W such that $V = U \oplus W$. Each G-module is a direct sum of irreducible submodules.

Theorem (Schur's Lemma): Let G be a group (compact or non-compact) and let V and W be irreducible G-modules. Then

(i) A morphism $f : V \to W$ is either zero or an isomorphism.

(ii) Every morphism $f : V \to W$ has the form $f(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.
(iii) $\dim_{\mathbb{C}} \mathrm{Hom}_G (V, W) = 1$ if $V \cong W$, and
$\dim_{\mathbb{C}} \mathrm{Hom}_G (V, W) = 0$ if $V \ncong W$.

Proof: (i) Since V is irreducible, the kernel of f is either $\{0\}$ or V. In the latter case f is zero, in the former f is injective. If f is injective, its image is a nonzero submodule of the irreducible G-module W, and hence is all of W. We conclude that f is an isomorphism, showing (i). To prove (ii), assume that f is nontrivial (nonzero) and let λ be any eigenvalue of f and W the corresponding eigenspace. Thus $W = \{w \in W \mid f(w) = \lambda w\}$ and one easily checks that W is a G-submodule. Hence $W = V$, which gives (ii). The third part follows from (i) and (ii).

Corollary: This is Tinkham's version of Schur's lemma: Any matrix which commutes with all matrices of an irreducible representation must be a constant matrix. Thus, if a nonconstant matrix exists, the representation is reducible.

Definition: If V and W are G modules, we may form their direct sum $V \bigoplus W$. This becomes a G-module with the action $g(v, w) = (gv, gw)$. In terms of matrices this corresponds to the following construction: If $G \to \mathrm{GL}(m, \mathbb{C}), g \mapsto A(g)$ and $G \to \mathrm{GL}(n, \mathbb{C}), g \mapsto B(g)$, then we obtain the direct sum representation $G \to \mathrm{GL}(m+n, \mathbb{C})$ by forming the block matrices

$$g \mapsto \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}$$

Proposition: An irreducible representation of an Abelian Lie group G (compact or non-compact) is one-dimensional.

Proof: This follows from Schur's lemma. Let V be an irreducible representation of an Abelian Lie group G. The translation $l_g : V \to V, v \mapsto gv$ is a linear morphism, since $l_g(hv) = ghv = hgv = hl_g(v)$ for any $h \in G$ (G Abelian). Therefore, l_g is a multiplication by a complex constant (Schur's lemma). For any vector $v \in V$, the one-dimensional subspace W generated by v is G-invariant. Therefore, W is a subspace of V, which implies that $W = V$, since V is irreducible.

Notation: Let G be a compact group. We denote by $\mathrm{Irr}(G, \mathbb{C})$ a complete set of pairwise nonisomorphic G-modules, i.e., each irreducible G-module is isomorphic to exactly one element of $\mathrm{Irr}(G, \mathbb{C})$. For arbitrary representations of G we use the following terminology:

If U is isomorphic to a submodule of V, we say that U is **contained** in V. If W is irreducible, we call $\dim_{\mathbb{C}} \mathrm{Hom}_G (W, V)$ the **multiplicity** of W in V. The significance of this number and its name is the following: Suppose we have a decomposition $V = \bigoplus_j V_j$ of V into irreducible submodules V_j. Then $\mathrm{Hom}_G (W, V) = \bigoplus_j \mathrm{Hom}_G (W, V_j)$, and so by Schur's lemma $\dim_{\mathbb{C}} \mathrm{Hom}_G (W, V)$ is simply the number of V_j that are isomorphic to W.

Theorem: For any G-module V, with G being a compact Lie group, there is a **canonical decomposition** into irreducible G-modules V_j, i.e.,

$$V = \bigoplus_j n_j V_j \tag{18.7}$$

with $V_j \in \mathrm{Irr}(G, \mathbb{C})$. The multiplicity n_j of a given submodule tells us how often it appears in the decomposition.

Proof: The condition "compact" is necessary. The existence of the decomposition follows from the existence of the G-invariant inner product. Simply choose a G-invariant orthonormal basis.

Definition: We have already introduced the direct sum of representations. Let us now study the tensor product. Let V and W be representations of G. The **tensor product** representation has the action

$$g(v \otimes w) = gv \otimes gw. \tag{18.8}$$

If $v_1, \ldots, v_n$ is a basis of V amd $w_1, \ldots, w_m$ a basis of W, with $gv_j = \sum_i r_{ij} v_i$ and $gw_l = \sum_k s_{kl} w_k$, then

$$g(v_j \otimes w_l) \;=\; \sum_{i,k} r_{ij} s_{kl} v_i \otimes w_k. \tag{18.9}$$

The matrix $(r_{ij} s_{kl})$ is sometimes called the **Kronecker product** of (r_{ij}) and (s_{kl}).

Exercises:

(X1) Show that an irreducible representation of a cyclic group is one-dimensional. (2 points)

19. ORTHOGONALITY RELATIONS

I hate indices and complicated formulas. Therefore, I just state the **great orthogonality theorem** in the following way. It is the same formula as in Tinkham's book, just without indices.

Theorem (Orthogonality relations): Let V be an irreducible unitary representation of a compact Lie group G. Then:

(i) For any G-morphism $f \in \mathrm{Hom}(V, V)$ and $v, w \in V$

$$\int_G \left\langle gf\left(g^{-1}v\right), w\right\rangle dg \;=\; \frac{1}{|V|}\mathrm{Tr}(f)\langle v, w\rangle; \quad \text{and} \tag{19.1}$$

(ii) For $v, w, \alpha, \beta \in V$

$$\int_G \langle \overline{g\alpha, v}\rangle \langle g\beta, w\rangle\, dg \;=\; \frac{1}{|V|}\left\langle \overline{\alpha, \beta}\right\rangle \langle v, w\rangle. \tag{19.2}$$

20. CHARACTERS

Definition: Given a compact Lie group G and a complex representation V, the **character** of V is the function

$$\chi_V : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(l_g), \tag{20.1}$$

where $\mathrm{Tr}(l_g)$ is the trace of the linear map $l_g : V \to V, v \mapsto gv$. If we just consider the matrix representation of the group, then the character χ_V is the function that assigns to a group element g the trace of the matrix associated with it. The character of an irreducible representation is called an **irreducible** character.

Given a family of representations $(V_j)_j$, we get a family of characters $(\chi_{V_j})_j$. These characters form a vector space (just like wave functions form a vector space), with the addition of functions being the vector addition and the multiplication by a constant being the multiplication with scalars. This vector space of characters has the following Hermitian product:

$$\langle \chi_i, \chi_j \rangle \;=\; \int_G \overline{\chi_j}\chi_i \;=\; \int_G \overline{\chi_j(g)}\chi_i(g)\; dg. \tag{20.2}$$

Proposition (Properties of the character function): Let χ_V be the character of a complex representation V of the compact Lie group G with neutral element e. Then we have for $g, h \in G$:

(i) χ_V is a C^∞-function.
(ii) If V and W are isomorphic, the $\chi_V = \chi_W$.
(iii) $\chi_V(ghg^{-1}) = \chi_V(h)$.
(iv) $\chi_{V \oplus W} = \chi_V + \chi_W$.
(v) $\chi_{V^*}(g) = \chi_V(g^{-1})$.
(vi) $\chi_{\overline{V}} = \overline{\chi_V(g)} = \chi_V(g^{-1})$.
(vii) $\chi_V(e) = \dim_{\mathbb{C}} V$.

We see that the character is constant on a conjugacy class of G. Such a function is called a **class function**.

The next theorem applies the great orthogonality theorem to characters. Since we mostly deal with characters, this is the formula you really have to understand. But first, a definition.

Definition: Let V be a complex representation of the compact Lie group G. Then the **fixed point space** of V is defined as

$$V^G \;=\; \{v \in V \,|\, gv = v \text{ for all } g \in G\}. \tag{20.3}$$

This fixed point space is a linear subspace of V. We have the canonical projection

$$p : V \to V^G, \quad v \mapsto \int_G gv\; dg \tag{20.4}$$

from the representation space into its fixed point space.

Theorem (Orthogonality Relations for Characters):

(i) $\int_G \chi_V(g)\, dg \;=\; \dim\; V^G$.
(ii) $\langle \chi_W, \chi_V \rangle \;=\; \int_G \overline{\chi}_V(g)\, \chi_W(g)\, dg \;=\; \dim\; \mathrm{Hom}_G(V, W)$.
(iii) The characters of the irreducible representations form an orthonormal basis of the vector space generated by all characters for a given group. In other words, if V and W are irreducible, then

$$\int_G \overline{\chi}_V(g)\,\chi_W(g)\;dg = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise.} \end{cases} \tag{20.5}$$

Since this formula is one of the most important in this course, I will write down explicitly what it means for a finite group: If V and W are irreducible, then

$$\frac{1}{h}\sum_{g\in G} \overline{\chi}_V(g)\,\chi_W(g) \;=\; \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise.} \end{cases} \tag{20.6}$$

Since the characters are constant within classes, we can write it in yet a different way: If V and W are irreducible, then

$$\frac{1}{h}\sum_{k} \overline{\chi}_V(g)\,\chi_W(g)\,N_k \;=\; \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{otherwise,} \end{cases} \tag{20.7}$$

where the sum now runs over all classes in the group and N_k is the number of elements in class k.

(iv) A representation is determined up to isomorphism by its character.

(v) If $\langle \chi_V, \chi_V \rangle$=1, then V is irreducible.

Proof:

(i) Let $V = \bigoplus_j n_j V_j$ be the canonical decomposition of V into irreducible representations V_j. Then $\chi_V = \sum_j n_j \chi_{V_j}$ and $n_j = \langle \chi_V, \chi_{V_j} \rangle$.

(ii) If $V = \bigoplus_j n_j V_j$, then $\langle \chi_V, \chi_V \rangle = \sum_j n_j^2 = 1$.

Theorem: Let G be a finite group. (i) The sum of the squares of the dimensions of the irreducible representations of G is equal to the number of elements in G. (ii) The sum of the squares of the characters in any irreducible representation equals the order of the group. a complex number z.)

21. CONJUGACY CLASSES AND THE REGULAR REPRESENTATION

Given a group G of order n with a finite number of elements, we can define the **group ring** $\mathbb{C}[G]$ (or **group algebra**), that is an abstract ring with elements of the form [1]

$$\sum_{g\in G} a_g g \tag{21.1}$$

with complex coefficients a_g. As a vector space, $\mathbb{C}[G]$ has the basis $\{g\}_{g\in G}$ consisting of all group elements. The product that makes $\mathbb{C}[G]$ an algebra (or ring) is defined as follows:

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) \;=\; \sum_{g,h\in G} (a_g b_h)(gh) \tag{21.2}$$

The group G is obviously embedded in the group algebra, since every element $h \in G$ can be written in the form $\sum_{g\in G} a_g g$ by setting $a_h = 1$ and $a_g = 0$ for $g \neq h$. The product thus defined is associative and has the unit element $e \in G$.

The **conjugacy class** of an element $g \in G$, that is the set of all elements $h \sim g$ (where $\sim$ denotes the equivalence relation of being conjugate), is also contained in the group ring as the element

$$\sum_{h\sim g} h. \tag{21.3}$$

I do not wish to go into too much mathematical trouble and therefore will not analyze the structure of the algebra $\mathbb{C}[G]$. However, I will note a number of very important theorems about characters following this structural analysis.

Theorem: Let G be a finite group. There are only a finite number of irreducible characters of G (over the field of complex numbers). The characters of all representations of G are the linear combinations of the irreducible characters with integer coefficients ≥ 0.

This may seem like a typical useless mathematical existence theorem. However, look at it this way: This theorem allows us to **list** all irreducible characters for the 32 point groups in a book with a finite number of pages. These characters are listed in Tinkham and other books. But the structure analysis tells us even more.

Theorem: The number of conjugacy classes of a finite group G of order n is equal to the number of irreducible characters. If d_i is the dimension of the i-th irreducible character χ_i and s the number of classes (or irreducible representations), then

$$\sum_{i=1}^{s} d_i^2 \;=\; n, \tag{21.4}$$

as we have proven in the section on the orthogonality relations for characters.

Theorem: The dimension d_i of the i-th irreducible representation divides the order of the group G.

Definition: Since $\mathbb{C}[G]$ is an n-dimensional vector space, we can view it as a representation of G (ord $G = n$), where the action of G on $\mathbb{C}[G]$ is defined in the obvious way:

$$G \times \mathbb{C}[G] \to \mathbb{C}[G]\,; \quad \left(h, \sum_{g\in G} a_g g\right) \mapsto \sum_{g\in G} a_g hg. \tag{21.5}$$

This representation is called the **regular representation**. The character of the regular representation

$$\chi_{\text{reg}} \;=\; \sum_{i=1}^{s} d_i \chi_i \tag{21.6}$$

is called the **regular character**. We can explicitly write the regular character as

$$\chi_{\mathrm{reg}}(g) \;=\; \begin{cases} 0 \;\text{ if } g \in G,\, g \neq e \in G \text{ and} \\ n \;\text{ if } g = e \in G. \end{cases} \tag{21.7}$$

This really helps, since it tells us how to construct the irreducible representations for any finite group: Simply construct the regular representation and then reduce it into irreducible components.

22. DIRECT-PRODUCT GROUPS

We have defined the direct product $G \times H$ of two groups G and H shortly after defining the groups and have shown that $G \times H$ is a group. We may therefore ask, what the irreducible representations of the direct product look like.

Theorem: If V is an irreducible representation of G and W is an irreducible representation of H, then $V \otimes W$ is an irreducible representation of $G \times H$. Furthermore, any irreducible representation of $G \times H$ is a tensor product of this form.

Proof: This is a consequence of the properties of the character and the orthogonality relations for the character.

23. SOME PRACTICAL TIPS ABOUT FINDING REPRESENTATIONS

Here is a situation you will find often: You have a give crystal structure (lattice and basis). Then you find the symmetry group for this crystal. You represent these symmetry operations by 3×3 matrices. Next you want to find all irreducible representations of the group. Here is how you can do it.

1. First, you try to write the group as a direct product of two other groups. If this works, you will get the irreducible representations of the direct product group as the tensor products of the irreducible representations of the factors in the direct product.
2. One method to find a way to write the group as a direct product is to look for normal divisors in the multiplication table, because you can factor them out. Remember that every subgroup of index two is normal. However, not every factor group gives you a way to write the full group as a direct product. Take, for example, the equilateral triangle group.
3. Assume now that you have worked your way down and that the group cannot be broken up into smaller pieces. Now find the irreducible representations of this group and the corresponding characters.
4. There is always the trivial representation. It maps all group elements onto the number 1. Its character is all 1s for all group elements.
5. There is always another one-dimensional representation, namely the determinant of the matrices. This representation will be the same as the trivial one, if there are no symmetry planes in the group.
6. Find the number of classes in the group. That tells you how many irreducible representations there are.

7. The dimension of any irreducible representation divides the order of the group.
8. The sum of squares of any irreducible character is equal to the order of the group.
9. Remember that the sum of squares of the dimensions of the irreducible representations is equal to the number of elements in the group.
10. Characters belonging to different irreducible representations are orthogonal.
11. The three-dimensional matrices give you a character, but it may not be irreducible. It is irreducible if and only if the sum of squares of the characters is equal to one.
12. If everything else fails: Construct the regular representation and then reduce it into its irreducible components.

24. CLASSICAL MECHANICS: POISSON BRACKETS

How can symmetry help us to solve problems in classical mechanics ? As you know, many-particle problems containing three or more particles are very difficult to solve anyway and two-particle systems can be separated into center-of-mass and relative coordinates. Therefore, we can confine our discussion to a single classical particle with coordinates $\vec{r}$ and momentum $\vec{p} = m\dot{\vec{r}}$. (More general treatments sometimes write $\vec{q}$ instead of $\vec{r}$ to denote a generalized coordinate.) The Hamiltonian function of this system is $H = T + V$, where T is the kinetic energy and V the external potential. Assume that we know the potential V and that we are looking for the complete solution to the **equations of motion**

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} \quad \text{and} \quad -\dot{\vec{p}} = \frac{\partial H}{\partial \vec{r}}. \tag{24.1}$$

How can symmetry help us ? (I assume here that you are familiar with the Hamiltonian formulation of the equations of motion and canonical transformations as outlined in chapters 7 and 8 of the Classical Mechanics book by Goldstein.) This discussion follows the book on *Nonrelativistic Mechanics* by Finkelstein [13].

We first define the classical commutators (**Poisson brackets**) of functions $u(\vec{r}, \vec{p})$ and $v(\vec{r}, \vec{p})$:

$$[u, v] = \frac{\partial u}{\partial \vec{r}} \cdot \frac{\partial v}{\partial \vec{p}} - \frac{\partial u}{\partial \vec{p}} \cdot \frac{\partial v}{\partial \vec{r}}. \tag{24.2}$$

The Poisson product $[-,-]$ makes the algebra of functions $u(\vec{r}, \vec{p})$ of $\vec{r}$ and $\vec{p}$ into a Lie algebra. The product is anticommutative, bilinear, and satisfies the Jacobi identity, therefore we have the following algebraic identities for dealing with Poisson brackets:

$$[u, v] = -[v, u], \tag{24.3}$$
$$[u, u] = 0, \tag{24.4}$$
$$[cu, v] = c[u, v] \quad \text{and} \tag{24.5}$$
$$[u, c] = 0 \quad \text{for any constant } c \text{ independent of } \vec{r} \text{ and } \vec{p}, \tag{24.6}$$
$$[u + v, w] = [u, w] + [v, w], \tag{24.7}$$
$$[uv, w] = u[v, w] + [u, w]v, \tag{24.8}$$
$$[u, vw] = [u, v]w + v[u, w], \tag{24.9}$$
$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0. \tag{24.10}$$

The **fundamental commutators** between coordinates and momenta are

$$[r_i, r_j] = 0, \tag{24.11}$$
$$[p_i, p_j] = 0, \tag{24.12}$$
$$[r_i, p_j] = \delta_{ij}, \tag{24.13}$$

just like you are used to from quantum mechanics. We can also see that

$$\frac{\partial u}{\partial \vec{r}} = [u, \vec{p}]\,, \tag{24.14}$$

$$\frac{\partial u}{\partial \vec{p}} = -\,[u, \vec{r}]\,, \quad \text{and} \tag{24.15}$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \tag{24.16}$$

for any function $u\,(\vec{r}, \vec{p}, t)$.

Using the Poisson brackets, we have a very elegant and symmetric way of writing the **equations of motion**:

$$[\vec{r}, H] \;=\; \dot{\vec{r}} \quad \text{and} \quad [\vec{p}, H] \;=\; \dot{\vec{p}}. \tag{24.17}$$

25. COORDINATE TRANSFORMATIONS

Let us now define **transformation operators** and their actions on the coordinates and momenta and functions of the coordinates and momenta. We start by defining some fundamental operators acting on time t, the coordinates $\vec{r}$, and the momenta $\vec{p}$:

- **Identity transformation:** The identity does nothing, that is

$$\mathrm{id} : \vec{r} \mapsto \vec{r}; \vec{p} \mapsto \vec{p}; t \mapsto t. \tag{25.1}$$

- **Translation of the spatial origin**: Let $T_{\vec{l}}$ be the operator that shifts all coordinates (but not the momenta) by a constant vector $\vec{l}$, that is

$$T_{\vec{l}} : \vec{r} \mapsto \vec{r} + \vec{l}; \vec{p} \mapsto \vec{p}; t \mapsto t. \tag{25.2}$$

  It is obvious that the set of all operators $\left\{ T_{\vec{l}} \,|\, \vec{l} \in \mathbb{R}^3 \right\}$ forms an Abelian Lie group isomorphic to $\mathbb{R}^3$ with the group operation $T_{\vec{l}} T_{\vec{l}'} = T_{\vec{l}+\vec{l}'}$. This group is non-compact, therefore its representations are somewhat difficult to deal with, since most theorems may not apply. However, since the group is isomorphic to $\mathbb{R}^3$ we may still be able to draw some conclusions, since we know the structure of $\mathbb{R}^3$ and its topology pretty well. It is obvious that three real parameters describe this group.

- **Translation of time**: By the same token, T_{t_0} is a translation in time:

$$T_{t_0} : t \mapsto t + t_{t_0}; \vec{r} \mapsto \vec{r}; \vec{p} \mapsto \vec{p}. \tag{25.3}$$

  Again, the set of all temporal translations $\{T_t \,|\, t \in \mathbb{R}\}$ is a non-compact Abelian Lie group isomorphic to $\mathbb{R}$ with the group operation $T_t T_t' = T_{t+t'}$.

- **Time reversal**: The name says it all. Note that the momentum changes sign under this operation.

$$\pi_t : t \mapsto -t; \vec{r} \mapsto \vec{r}; \vec{p} \mapsto -\vec{p}. \tag{25.4}$$

- **Spatial inversion** (or parity):

$$\pi : \vec{r} \mapsto -\vec{r}; \vec{p} \mapsto -\vec{p}; t \mapsto t. \tag{25.5}$$

- **Proper rotations**: For all matrices $A \in \mathrm{SO}(3)$, we have the operation

$$A : \vec{r} \mapsto A\vec{r}; \vec{p} \mapsto A\vec{p}; t \mapsto t. \tag{25.6}$$

  We identify this group of transformations with the matrix group $\mathrm{SO}(3)$. This group has three parameters, that is Euler's angles.

- **Reflections:** As an example for a reflection in a plane containing the origin, assume that the plane contains the x- and y-axes. The reflection operator acts on the coordinates in the following way:

$$\sigma : \vec{r} = (x, y, z) \mapsto (x, y, -z); \vec{p} = (p_x, p_y, p_z) \mapsto (p_x, p_y, -p_z); t \mapsto t. \tag{25.7}$$

  Reflections can be described by matrices with determinant -1, therefore they do not form a group.

- **Composite transformations:** We can also study composite transformations that combine a number of these basic operations. We note that all compositions of the transformations listed above yield a transformation of the form

$$R : (\vec{r}, \vec{p}, t) \mapsto \left(A\vec{r} + \vec{l}, A\vec{p}, t + t_0\right) \tag{25.8}$$

  with a matrix A, a vector $\vec{l}$ and a real number t_0. In order to parametrize this symmetry group, we need three parameters for spatial translation, three Eulerian angles for rotation, one for parity, one for time-reversal, and one for temporal translations. The group formed by all spatial rotations and translations is known as the **Euclidean** group. Examples: An **improper rotation** is a rotation followed by inversion or a reflection. Another example from high-energy physics is CPT symmetry. This operator combines charge conjugation (C), parity (P), and time reversal (T). Some crystal classes have a glide plane or screw axis (reflection or rotation followed by a translation) as a symmetry element.

- **Inverse transformation:** All these coordinate transformations R have an inverse transformation R^{-1} that undoes the change of the coordinates, that is $RR^{-1} = \mathrm{id}$.

$$R^{-1} : (\vec{r}, t) \mapsto \left(A^{-1}\left(\vec{r} - \vec{l}\right), A^{-1}\vec{p}, t - t_0\right) \tag{25.9}$$

  This is obvious for the transformations that form matrix groups, where the inverse of the transformation is given by the inverse matrix. For reflections or inversions, the inverse is identical to the transformation.

- **Galilean transformation:** The Galilean transformation corresponds to a change of reference frame, for example from the lab frame to the center-of-mass frame. We will assume that this change can be done non-relativistically. This can be thought of as a translation in momentum space, where the translation vector is given by $m\vec{v}$, if m is the mass of the particle. In terms of equations:

$$T_{m\vec{v}} : \vec{r} \mapsto \vec{r} + t\vec{v}; \vec{p} \mapsto \vec{p} + m\vec{v}; t \mapsto t. \tag{25.10}$$

The group formed by these reference frame transformations and by translations in time is called the **Galilean** group.

- **Ten-parameter group:** The direct-product group of all Euclidean and Galilean transformations has ten parameters. In addition to these ten parameters, we also have parity and time-reversal symmetry.

We now have to consider how these symmetry operators **induce** operators acting on functions containing time t and the coordinates $\vec{r}$ (such as the Hamiltonian function H). In order to simplify our notation, we will soon use the same symbols for the operators acting on the coordinates and the operators acting on functions of the coordinates (unless there might be an ambiguity), but we really have to remember that these two classes of operators are different things, although the groups they form are isomorphic. There are actually two ways how we can define the action of a transformation on a function. They are sometimes called **active** and **passive** conventions. In the active convention, we rotate the object, in the passive convention, we rotate the axes of the coordinate system and leave the object fixed. In my intuition, I tend to use the active (object-rotation) convention. For functions instead of objects, the whole situation is a little different, but there are still two different ways to define these things. Here, we follow Wigner's convention (as in Tinkham's book):

For a coordinate transformation $R : (\vec{r}, \vec{p}, t) \mapsto \left(A\vec{r} + \vec{l}, A\vec{p}, t + t_0\right)$ acting the coordinates $\vec{r}$, momenta $\vec{p}$, and time t we define the operator P_R acting on a function of $\vec{r}$, $\vec{p}$, and t in the following way:

$$P_R : f \mapsto P_R f, \quad \text{with} \quad P_R f \quad \text{defined by} \tag{25.11}$$

$$P_R f : (\vec{r}, \vec{p}, t) \mapsto P_R f\left(\vec{r}, \vec{p}, t\right) = f\left(R^{-1}\left(\vec{r}, \vec{p}, t\right)\right) = f\left(A^{-1}\left(\vec{r} - \vec{l}\right), A^{-1}\vec{p}, t - t_0\right). \tag{25.12}$$

$$\text{Therefore} \quad P_R f\left(R\left(\vec{r}, \vec{p}, t\right)\right) = f\left(\vec{r}, \vec{p}, t\right). \tag{25.13}$$

In other words, P_R changes the functional form of $f\left(\vec{r}, \vec{p}, t\right)$ in such a way as to compensate for the change of variables caused by R. These operators form a group with the operation $P_R P_S = P_{RS}$. In a similar way, we define operators acting on functions induced by symmetry transformations in the Galilean group.

26. SYMMETRY AND CLASSICAL MECHANICS

Now let us consider the group of operators P_R which commute with the Hamiltonian. (In quantum mechanics, we will call this the **group of the Schrödinger equation**, but let us call it the **group of the Hamiltonian** for now.) These operators P_R do form a group, because if two operators commute with H, their product also commutes with H. The coordinate transformations R whose operators P_R commute with H are called symmetry operations.

Wait a minute ! That does not work ! The elements of the symmetry groups are operators, and the Hamiltonian is a function of $\vec{r}$, $\vec{p}$, and t. We cannot define a commutator between an operator and a function, since they are different types of mathematical objects. Quickly forget about the statements in the paragraph above. We have to find a different formalism for these things. As you read on (and compare with other books) I hope you will appreciate the lack of indices and summations. I agree, however, that things tend to be a little abstract. (I call this geometry.)

Before we go on to discuss the general case, let us study a specific example, the connection between momentum conservation and translational invariance. A translation by $\vec{l}$ leaves a function invariant, if $\left(T_{\vec{l}} - 1\right) f = 0$. For small $\vec{l}$, we find by Taylor-expansion

$$\left(T_{\vec{l}} - 1\right) f(\vec{r}, \vec{p}, t) \;=\; -\frac{df}{d\vec{r}} \cdot \vec{l} \;=\; -[f, \vec{p}] \cdot \vec{l} \tag{26.1}$$

We have found that a translation $T_{\vec{l}}$ leaves the Hamiltonian invariant if and only if the Hamiltonian commutes with the momentum $\vec{p}$, which means that the momentum is conserved, since

$$\frac{d\vec{p}}{dt} \;=\; [\vec{p}, H] + \frac{\partial \vec{p}}{\partial t} \;=\; [\vec{p}, H]\,, \tag{26.2}$$

as we have seen before. More general: A translation in a certain direction leaves the Hamiltonian invariant, if the Hamiltonian commutes with the momentum component in that direction. This means that the component of momentum in that direction is conserved. This is a well-defined statement, much better than the one in the first paragraph of the chapter. Because of this relationship between translational invariance and momentum, the momentum is called the **infinitesimal generator** of translations.

Now it is time to generalize what we have learned above and to refer to the concepts of the Lie algebra and the exponential map from the mathematical sections. The elements of the Lie algebra are called **infinitesimal generators**, since they generate the Lie groups made up by Euclidean and Galilean transformations. The exponential map connects the Lie algebra with the Lie group, as defined in detail in the mathematical chapters. The Lie algebra is locally isomorphic and homeomorphic to (for the physicists: locally looks like) a sufficiently small neighborhood of the neutral element in the Lie group. Since this neighborhood generates the whole group (if the group is connected), the structure of the Lie algebra and the exponential map are sufficient to describe the structure of the Lie group. This does not really help us with the group of translations (other than showing the connection between translational invariance and momentum conservation), but there are more complicated groups (such as rotations) where this will really make things easier.

Let us start with a simple example (translational invariance) for the translation group G isomorphic to $(\mathbb{R}^3, +)$ consisting of translations $T_{\vec{l}}$. The Lie algebra LG is also isomorphic to $\mathbb{R}^3$. For each vector $\vec{l} \in \mathbb{R}^3$, that is for each translation $T_{\vec{l}}$, we have the homomorphism of groups (called **one-parameter group**)

$$\alpha_{\vec{l}} : \mathbb{R} \to G = \left(\mathbb{R}^3, +\right), \quad t \mapsto \alpha_{\vec{l}}(t) = t\vec{l}. \tag{26.3}$$

This one-parameter group $\alpha_{\vec{l}}$ corresponds to the **tangent vector** (directional derivative) $X_{\vec{l}}$ in the Lie algebra $LG = L\mathbb{R}^3 = \mathbb{R}^3$ acting on a germ $\phi : G = \mathbb{R}^3 \to \mathbb{R}$ in the following way

$$X_{\vec{l}} : \mathcal{E}_{\vec{0}} \to \mathbb{R}, \quad \phi \mapsto X_{\vec{l}}(\phi) = \left.\frac{\partial}{\partial t}\right|_{t=0} \phi\left(\alpha_{\vec{l}}(t)\right) \;=\; \left.\frac{\partial}{\partial t}\right|_{t=0} \phi\left(t\vec{l}\right) \;=\; \vec{l}\cdot \mathrm{grad}\ \phi\left(\vec{0}\right) \;=\; \vec{l}\cdot [\phi, \vec{p}]\,, \tag{26.4}$$

where $\mathcal{E}_{\vec{0}}$ is the set of all germs of functions $\phi : \left(G, \vec{0}\right) \to \mathbb{R}$. We may assume that $\vec{l}$ is small in this case, since ϕ is only a germ, that is defined only in a small neighborhood of $\vec{0}$. It is obvious that $\dot{\alpha}_{\vec{l}}(0) = \vec{l}$, therefore we do have the right one-parameter group.

The **exponential map** is given as follows:

$$\exp : LG = \mathbb{R}^3 \to G = \mathbb{R}^3 \cong \left\{T_{\vec{l}}\,|\,\vec{l} \in \mathbb{R}^3\right\}, \quad \vec{l} \mapsto \alpha_{\vec{l}}(1) = \vec{l} \cong T_{\vec{l}}. \tag{26.5}$$

In other words, the exponential map is the identity of $\mathbb{R}^3$. Surprised ? Well, since everything seems to be isomorphic to $\mathbb{R}^3$, any other result would have been very significant. I should note again that the exponential map defines a local diffeomorphism between the Lie algebra and the Lie group (locally, these neighborhoods are the same). Therefore, there is an inverse map, which is also a local diffeomorphism. We have therefore found out how to calculate the **logarithm** of a group element:

$$\ln : G = \mathbb{R}^3 \to LG = \mathbb{R}^3, \quad T_{\vec{l}} \cong \vec{l} \mapsto \ln\left(T_{\vec{l}}\right) \;=\; \vec{l}. \tag{26.6}$$

How can we generalize these results about translational invariance to other symmetry properties ? Given a symmetry group G consisting of symmetry elements that leave the Hamiltonian invariant, we pick those infinitesimal symmetry transformations R which are different from the identity by only a small amount and study how they transform arbitrary functions $f(\vec{r}, \vec{p}, t)$. Let $l = \ln\ R$ be defined as the logarithm of the group element R, that is $\exp(l) = R$. Then we try to find an infinitesimal generator p such that

$$(R-1)\,H \;=\; -\,[H,p]\,l. \tag{26.7}$$

From this we find that

$$(R-1)\,H \;=\; -\,[H,p]\,l \;=\; -\frac{dp}{dt}l. \tag{26.8}$$

Therefore, if the symmetry operation R leaves the Hamiltonian invariant, then the infinitesimal generator p is conserved.

I note that this formalism only applies to connected Lie groups, not for finite groups. Therefore, this formalism will not help us with parity, time reversal, or crystal lattice symmetries, since they cannot be generated by infinitesimal symmetry operations.

It is not obvious how to find the infinitesimal generators, since this is a consequence of the definition of the Lie algebra structure (Poisson brackets). This is therefore an algebraic problem, not a geometrical one, and cannot be solved using group theory. The inifinitesimal generators are also called **integrals of motion**.

To summarize what we have learnt above: [14] Assume that there is a particle experiencing no external force. Then the Hamiltonian describing the motion of the particle does not depend on the coordinates. (The universe is **homogeneous**.) Therefore, the Hamiltonian is invariant with respect to translations. We look for the infinitesimal generators for translations and find momentum. Therefore, momentum is conserved for this mechanical system.

Another example: We have a closed system containing a particle moving in a potential without any outside forces. Therefore, the Hamiltonian of this system does not depend on time, the Hamiltonian is invariant with respect to temporal translations. The logarithm corresponding to a temporal translation T_t is t, therefore the infinitesimal generator B we are looking for has to satisfy the condition

$$(T_t - 1)\, H \;=\; -\frac{dH}{dt} t \;=\; -[H, B]\, t. \tag{26.9}$$

Since the Hamiltonian usually does not explicitly depend on time, the Hamiltonian itself is the infinitesimal generator, and temporal invariance implies conservation of energy. We say that time is **homogeneous**. (If the potential depends on the momentum coordinates, for example for non-conservative frictional forces, then the statements above are not true.)

Let us now assume that space is **isotropic**, that is a rotation of the coordinate axes does not change the Hamiltonian. If R is a small rotation by an angle $d\omega$ and the rotation axis points into the direction of the vector $d\vec{\omega}$, then

$$(R - 1)\, H \;=\; -d\vec{\omega} \cdot \left[H, \vec{L}\right] \;=\; 0. \tag{26.10}$$

Therefore, rotational invariance implies conservation of angular momentum. It would be interesting to study the details of this conservation law, the Lie algebra structure, and the exponential map for this case, but I don't have time right now.

The case of Galilean transformations is a bit more difficult, since the Hamiltonian is not invariant under a Galilean transformation. (The Hamiltonian is not a scalar under Galilean transformations. Quite obviously, the kinetic energy changes. H is one component of the energy-momentum vector under a Lorentz transformation. Maybe a relativistic formalism would straighten these things out.) However, the equations of motion are unchanged when transforming from one intertial frame to another. Nevertheless we can study the effect of a small Galilean transformation $T_{m\vec{v}}$ on the Hamiltonian:

$$(T_{m\vec{v}} - 1)\, f \;=\; -\left[f, \vec{G}\right] \cdot \vec{v} \tag{26.11}$$

The infinitesimal generator G for Galilean transformations is given by

$$\vec{G} \;=\; m\vec{r} - t\vec{p}. \tag{26.12}$$

If the system is free and momentum is conserved, we notice that

$$\frac{d\vec{G}}{dt} \;=\; m\frac{d\vec{r}}{dt} - \vec{p} \;=\; 0, \tag{26.13}$$

therefore $\vec{G}$ is conserved for a free system. The physical meaning of this conservation law is that an isolated system moves uniformly in a straight line. (For a system with many particles, this means that the center of mass moves uniformly in a straight line.) We see that there may be conservation laws in classical mechanics related to symmetry transformations that do not leave the Hamiltonian invariant.

Putting everything together, we see that the combined Euclidean and Galilean symmetry operations form a ten-parameter symmetry group. The Lie algebra of this symmetry group is formed by ten infinitesimal generators X_k. The structure of the algebra is defined by the commutators (Lie product) of the generators

$$[X_k, X_l] = \sum_m C_{klm} X_m = C_{klm} X_m. \tag{26.14}$$

For the rest of this chapter, we assume that we have to sum over repeated indices. The numbers C_{klm} are called the **structure constants** of the Lie algebra. They are time-independent. For the ten-parameter group, they are given in the following table:

$C_{klm}X_m$	P_l	G_l	L_l	H
P_k	0	$-m\delta_{kl}$	$\epsilon_{klm}P_m$	0
G_k			$\epsilon_{klm}G_m$	P_k
L_k			$\epsilon_{klm}L_m$	0
H				0

The constants ϵ_{klm} make up the totally antisymmetry tensor or rank three.

You will note that the commutator $[P_k, G_l]$ does not satisy the condition (26.14). The corresponding representation is called projective. A fully relativistic treatment changes this bracket and straightens things out. Isn't is nice to see that classical mechanics does not always work ? Even for small velocities, we can see that the theory is not consistent.

In the theory of special relativity, coordinates and time as well as energy and momentum form four-vectors. Four-dimensional translational invariance gives us the conservation of the energy-momentum four-vector. The group of rotations acting on the space/time coordinates is the four-dimensional rotation group $SO(4)$. Its Lie algebra is six-dimensional, and the three components of $\vec{L}$ together with those of $\vec{G}$ form the infinitesimal generators.

In special relativity, invariance of a system under four-dimensional translations implies conservation of four-dimensional energy-momentum, and invariance of a system under four-dimensional rotations implies conservation of the six-dimensional angular momentum. This symmetry group is called the **Lorentz group** or **Poincarè group**.

Structure constants are very important when studying high-energy group-theory, with its variety of symmetry groups. A Lie algebra is called **simple**, if it does not contain an **invariant subalgebra**, that is some set of generators which goes onto itself (or zero) under commutation with any element of the algebra. A Lie algebra without any Abelian subalgebras is called **semisimple**.

Exercises:

(X1) Find how H transforms under Galilean transformations.

(X2) Starting from the relativistic formulation of classical mechanics, using the relativistic Hamilton function

$$H = c\sqrt{p^2 + m^2c^2} + V \tag{26.15}$$

for a particle with momentum p and mass m in a potential V, find the infinitesimal generators of the Lorentz group and calculate the structure constants of the corresponding Lie algebra. (You may refer to the second chapter of the second volume of the series by Landau and Lifshitz [15] - or other books - for help.)

27. REPRESENTATIONS IN CLASSICAL MECHANICS

Let's start with the equations of motion:

$$[\vec{r}, H] = \dot{\vec{r}} \quad \text{and} \quad [\vec{p}, H] = \dot{\vec{p}}. \tag{27.1}$$

These equations are linear, therefore linear combinations of solutions $(\vec{r}_1, \vec{p}_1)$ and $(\vec{r}_2, \vec{p}_2)$ are also a solution. In other words, the solutions form a vector space. We can call this solution space a Hilbert space. For simplicity, I will use only $\vec{r}$ to denote a solution in the future, and I assume that it is obvious which $\vec{p}$ is supposed to go with $\vec{r}$. Well, maybe it is not obvious, but at least there is unique solution $\vec{p}$ to go with each $\vec{r}$ for a given set of initial or boundary conditions.

Now assume that R is a symmetry operation that leaves H invariant, that is

$$P_R H(\vec{r}, \vec{p}, t) = H(\vec{r}, \vec{p}, t). \tag{27.2}$$

Therefore, by applying the operator P_R to all functions, we note that

$$[P_R\vec{r}, H] = P_R\dot{\vec{r}} \quad \text{and} \quad [P_R\vec{p}, H] = P_R\dot{\vec{p}}. \tag{27.3}$$

We see that $P_R\vec{r}$ and $P_R\vec{p}$ also form a solution to the equations of motion.

By defining an action

$$G \times V \to V \quad (R, (\vec{r}, \vec{p})) \mapsto (P_R\vec{r}, P_R\vec{p}), \tag{27.4}$$

we see that every vector space of solutions $(\vec{r}, \vec{p})$ forms a representation with the action defined above, **if** it is closed under the elements of G, that is

$$(P_R\vec{r}, P_R\vec{p}) \in V \quad \text{for all} \, (\vec{r}, \vec{p}) \in V, R \in G. \tag{27.5}$$

In practive, given one solution of the equations of motion, we can generate others by applying the operators P_R for all group elements $R \in G$ to the given solution. However, the representations can be much more useful. Let us go through the symmetry groups one by one:

A. Parity

The solutions belonging to the irreducible representations of spatial inversion π are the even and odd solutions. However, all non-trivial solutions are odd, if inversion is a symmetry of the system, since $-\vec{r} = \pi\vec{r}$ by definition. This tells us that not all irreducible representations have to appear in nature. This result is somewhat surprising, but it is inherent in the laws of classical mechanics. We use $\vec{r}$ both as a coordinate, and also as the basis for describing the symmetry element.

If inversion is a symmetry of the system, then $\vec{r}(t)$ and $\pi\vec{r}(t) = -\vec{r}(t)$ will both be solutions of the equations of motion. Therefore, all the solutions are odd.

In quantum mechanics, of course, the case would be different. A wave function could be odd or even, depending on the phase, but a classical particle does not have a phase, therefore all solutions are odd. (Well, odd and even are used in different terms in classical mechanics and quantum mechanics.)

What happens if spatial inversion is not a symmetry of the system ? Well, $-\vec{r} = \pi\vec{r}$ is still true, but $\vec{r}$ and $-\vec{r}$ are not necessarily both solutions to the equations of motion.

B. Time-reversal

For time-reversal symmetry, the situation is different: If a system has time-reversal symmetry, then both $\vec{r}(t)$ and $\pi_t\vec{r}(t) = \vec{r}(-t)$ are solutions. Now, we can distinguish between even solutions $\vec{r}(t) = \vec{r}(-t)$ and odd solutions $\vec{r}(t) = -\vec{r}(-t)$. Since any representation can be broken up into irreducible components, any solution can be written as a sum of even and odd solutions. However, not every solution has to be odd or even, it simply has to split into even and odd components. Since all irreducible representations are one-dimensional, we can usually not generate new solutions from a given one.

What do we learn from this for solving classical mechanics problems ? Symmetry allows us to make an *Ansatz*, that is we can write any solution as a sum of solutions with certain properties. This will make it easier, to solve the Hamiltonian.

C. Spatial Translations

Translations, either in time or space or Galilean transformations: I wish I could help you here. The problem is that the translation group is not compact, therefore I do not know what the representations are. It would be interesting to study the representations of the group $(\mathbb{R}, +)$, but I don't know how to deal with this. There is one way how to handle this, though. We do what we always do when we can't get any further: **Introduce periodic boundary conditions.** Put the particle in a box. This reduces the translation group $\mathbb{R}^3$ to the **torus** $T^3 = \mathbb{R}^3/\mathbb{Z}^3 = (\mathbb{R}/\mathbb{Z})^3 = (S^1)^3$.

The torus T^n is a compact Abelian Lie group, therefore all its irreducible representations are one-dimensional, i.e., the representations and the characters are the same. We have the following theorem: The irreducible complex characters of the torus T^n have the form

$$\vartheta_\alpha : T^n \to \mathbb{R}, \quad [l] \mapsto \exp(2\pi i \alpha(l)), \quad l \in \mathbb{R}^n, \tag{27.6}$$

with $\alpha(l) = \langle a, l \rangle = \sum_\nu a_\nu l_\nu$, $a = (a_1, ..., a_n) \in \mathbb{Z}^n$. In the one-dimensional case $n = 1$, the representation is given by

$$\vartheta_\alpha : T^1 \times \mathbb{C} \to \mathbb{C}, \quad ([l], z) \mapsto z \exp(2\pi i \alpha l). \tag{27.7}$$

For any integer $\alpha \in \mathbb{Z}$ we get one representation. For $\alpha = 0$ we have the trivial representation doing nothing (multiplication by one). For nonzero α, the torus gets wrapped around α times, either clockwise or counterclockwise depending on the sign of α.

The Lie algebra of the torus is $LT^n = \mathbb{R}^n$, and the exponential map goes like this

$$\mathbb{R}^n = LT^n \to T^n, \quad l \mapsto [l] \tag{27.8}$$

In the three-dimensional (physics) case, we replace α by $\vec{k}$, absorb the factor 2π in the **wave vector** $\vec{k}$, and rewrite the characters to conform to our usual notation. Then, each $\vec{k} \in 2\pi\mathbb{Z}^3$ gives us a character

$$\vartheta_{\vec{k}} : T^3 \to \mathbb{R}, \quad \left[\vec{l}\right] \mapsto \exp\left(i\vec{k} \cdot \vec{l}\right), \quad \vec{l} \in \mathbb{R}^3. \tag{27.9}$$

What have we learnt from this ? A solution $\vec{r}(t)$ of the classical equations of motion belonging to the irreducible representation labelled by $\vec{k}$ transforms as

$$T_{\vec{l}}\vec{r}(t) \;=\; \vec{r}(t) + \vec{l} \;=\; \exp\left(i\vec{k} \cdot \vec{l}\right) \vec{r}(t) \tag{27.10}$$

This means that shifting the orbit $\vec{r}(t)$ of the particle by a constant vector $\vec{l}$ causes multiplication by an oscillatory phase term, therefore the orbit $\vec{r}$ must be periodic (plane wave).

Any solution $\vec{r}(t)$ to the equations of motion can be written as a Fourier series (that is, as a sum of plane waves)

$$\vec{r}(t) \;=\; \sum_{\vec{k} \in 2\pi\mathbb{Z}^3} c\left(\vec{k}\right) \exp\left(i\vec{k} \cdot \vec{u}\right). \tag{27.11}$$

Our intuition (call it experience with physics) tells us that the representations of the non-compact group $\mathbb{R}^3$ have the same form as the representations of the torus, but now $\vec{k}$ can be any vector in $\mathbb{R}^3$ and need not be an integral multiple of 2π. Therefore, any solution of a system with translational invariance can be written as

$$\vec{r}(t) \;=\; \int d\vec{k}\; c\left(\vec{k}, t\right) \exp\left(i\vec{k} \cdot \vec{r}\right). \tag{27.12}$$

This tells us that any solution to the equations of motion of a system with translational invariance can be expanded into plane waves.

Now it hurts that we did not introduce **real representations**, only complex ones. However, our intuition tells us that in the real case of classical mechanics, we get two real irreducible representations for each complex one, [10] namely $\sin\left(2\pi\vec{k}\cdot\vec{r}\right)$ and $\cos\left(2\pi\vec{k}\cdot\vec{r}\right)$. You may have encountered this hand-waving argument in your first physics class, but how do you explain real and complex representations to a freshman ?

D. Translations in Time

If the Hamiltonian does not explicitly depend on time (if it is invariant to temporal translations), then the solutions to the equations of motion can be written as a sum (if we introduce periodic boundary conditions in time, that is make the process periodic)

$$\vec{r}(t) = \sum_k c_k(\vec{r}) \exp(ikt). \tag{27.13}$$

We already know this. A periodic process can be expanded into a Fourier series. If we drop the periodic boundary conditions, then the solution can be written as a Fourier integral.

E. Crystal Symmetry

Study the case of a classical particle in a periodic potential. There is no translational invariance here, at least arbitrary or infinitesimal translations do not leave the Hamiltonian invariant. However, a translation by a vector $T = n_1\vec{a}_1 + n_2\vec{a}_2 + n_3\vec{a}_3$ with any combination of integers $(n_1, n_2, n_3) \in \mathbb{Z}^3$ does not change the Hamiltonian. This symmetry group is isomorphic to $\mathbb{Z}^3$, not compact, and therefore very difficult to deal with. Therefore, we introduce periodic boundary conditions and reduce the symmetry group to a cyclic group $(\mathbb{Z}/n)^3$. The irreducible character of $\mathbb{Z}/n$ are given by the n functions

$$\mathbb{Z}/n \to \mathbb{R} \quad [l] \mapsto \exp(2\pi i l j/n), \quad j = 0, 1, \ldots, n-1. \tag{27.14}$$

Therefore, any solution of the classical equations of motion can be written as a sum

$$\vec{r}(t) = \sum_{\vec{k}} c\left(\vec{k}, t\right) \exp\left(i\vec{l}\cdot\vec{k}\right), \tag{27.15}$$

where we substituted $k = 2\pi j/n$. This is known as **Bloch's theorem**. As you can see, we have not used quantum mechanics or the Schrödinger equation in any way. Therefore, this formalism can also be applied to a classical treatment of lattice vibrations.

For an infinite crystal, we have

$$\vec{r}(t) = \int d\vec{k}\; c\left(\vec{k}, t\right) \exp\left(i\vec{l}\cdot\vec{k}\right), \tag{27.16}$$

where the summation runs over all

Exercises:

(X1) What are the conjugacy classes of the torus T^1 ? What are the conjugacy classes of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ (all with addition as the law of composition) ?

28. GROUP THEORY AND QUANTUM MECHANICS

In quantum mechanics, we can define the group of the Schrödinger equation, or the group of the Hamiltonian, as the group G of all symmetry operations R whose operators P_R commute with the Hamiltonian H. For any wave function ψ_n with energy eigenvalue E_n, we have

$$H\ P_R\ \psi_n \ =\ P_R\ H\ \psi_n \ =\ P_R\ E_n\ \psi_n \ =\ E_n\ P_R\ \psi_n. \tag{28.1}$$

Therefore, $P_R\psi_n$ is also a wave function in the same eigenspace with energy eigenvalue E_n. Let us call this eigenspace V. We can define an action of the group G on V

$$G \times V \to V \quad (R, \psi) \ \mapsto\ P_R\psi \tag{28.2}$$

which makes V a representation of G. We have now made the connection between quantum mechanics and the theory of representations. Is this representation irreducible ? It depends: If there are no accidental degeneracies, then all wavefunctions in V can be generated from any one of them by applying all elements of the group. In this case, the degeneracy of the energy eigenvalue is called **normal**, and the representation is irreducible, since there are no G-invariant subspaces. If the degeneracy is not normal, there are some eigenfunctions which cannot be generated from the one you start with. Therefore, there is a G-invariant subspace, and the representation is reducible. In this case, there is usually some hidden additional symmetry in the system.

We may assume that this representation is unitary, since we can find a G-invariant inner product of V and then choose a G-orthonormal basis of eigenfunctions. Therefore, the matrices appearing in the representation may assumed to be unitary matrices, that is matrices whose inverse is identical to the conjugate transpose.

We summarize: If the degeneracy of an energy eigenvalue is normal, then the eigenspace for any energy eigenvalue forms an irreducible representation of the group of the Hamiltonian. Apparently, this imposes some restrictions on what kind of degeneracies there can be.

Therefore, by classifying the irreducible representations of a given symmetry group, we can determine the degrees of (nonaccidental) degeneracies that can appear in any quantum mechanical problem. This implies that a perturbation of the system can lift degeneracies only if its inclusion in the Hamiltonian reduces the symmetry group and therefore changes the possible irreducible representations. In solid-state physics, this is called **crystal-field splitting**.

I should also mention that a set of pairwise commuting operators can be diagonalized simultaneously. The eigenvalues belonging to these operators are called "good quantum numbers". We conclude that any symmetry operator which commutes with the Hamiltonian generates a set of good quantum numbers. If we can find the complete symmetry of the system, we can find a maximal set of pairwise commuting operators.

29. BASIS FUNCTIONS

We have seen that the orthonormal eigenfunctions spanning each energy eigenspace of the Hamiltonian of a quantum mechanical system form an **irreducible unitary** representation of the symmetry group of the system. These eigenfunctions are wave functions, of course, and describe a possible state of the system. The functions forming a basis of the eigenspace are called **basis functions** and labelled with two labels: We need one label for the irreducible representation, and a second one to label the particular function within the representation. All basis functions belonging to the same irreducible representation are called **partners**.

These basis functions form a complete set of all wave functions, that is any wave function can be written as a sum

$$\psi\left(\vec{r}\right) = \sum_{j=1}^{n} \sum_{\kappa=1}^{l_j} \psi_\kappa^{(j)}\left(\vec{r}\right), \tag{29.1}$$

where j labels the irreducible representations, n is the number of irreducible representations (this can be infinity), l_j is the dimension of the j-th representation, and $\psi_\kappa^{(j)}$ is the κ-th partner function belonging to the κ-th row of the j-th irreducible representation.

According to the definition of the matrix representation by matrices $\Gamma^{(j)}\left(g\right)_{\lambda\kappa}$, we see that the group element g defines an operator P_g acting on the κ-th basis function of the j-th irreducible representation as follows:

$$g\phi_\kappa^{(j)} = P_g\phi_\kappa^{(j)} = \sum_{\lambda=1}^{l_j} \Gamma^{(j)}\left(g\right)_{\lambda\kappa}\ \phi_\lambda^{(j)} \tag{29.2}$$

Given any function ψ, we may want to obtain the component of ψ belonging to the λ-th partner of the j-th irreducible component. This is done with the operator

$$\mathcal{P}_{\lambda\kappa}^{(j)} = l_j \int_G dg\ \overline{\Gamma^{(j)}\left(g\right)_{\lambda\kappa}} P_g = \frac{l_j}{h} \sum_g \overline{\Gamma^{(j)}\left(g\right)_{\lambda\kappa}} P_g \tag{29.3}$$

and uses the orthogonality relations for the representation matrices

$$\int_G dg\ \overline{\Gamma^{(j)}\left(g\right)_{\lambda\kappa}} \Gamma^{(i)}\left(g\right)_{\mu\nu} = \frac{1}{l_j}\delta_{ij}\delta_{\lambda\mu}\delta_{\kappa\nu} \tag{29.4}$$

and the fact that the basis functions form an orthonormal basis. We can see that

$$\mathcal{P}_{\lambda\kappa}^{(j)}\phi_\nu^{(i)} = 0 \quad \text{for } i \neq j \text{ or } \kappa \neq \nu \text{ and} \tag{29.5}$$

$$\mathcal{P}_{\lambda\kappa}^{(j)}\phi_\kappa^{(j)} = \phi_\lambda^{(j)}. \tag{29.6}$$

In words, this means that when the operator $\mathcal{P}_{\lambda\kappa}^{(j)}$ is applied to a basis function, then the result is zero, unless the basis function belongs to the κ-th row of the j-th irreducible representation. If the basis function does satisfy this condition, then we obtain its partner belonging to the λ-th row of the same representation. Van Vleck called this operator the **basis-function generating machine**, since it allows us to generate all partner functions in a given irreducible representation from one of the partners.

We can also study the projection operators $\mathcal{P}^{(j)}_{\kappa\kappa}$ and

$$\mathcal{P}^{(j)} \;=\; \sum_{\kappa} \mathcal{P}^{(j)}_{\kappa\kappa} \;=\; l_j \int_G dg\, \overline{\chi_j(g)} P_g \tag{29.7}$$

The projection operator $\mathcal{P}^{(j)}_{\kappa\kappa}$ projects the component belonging to the κ-th partner of the j-th irreducible representation out of any given wave function, whereas $\mathcal{P}^{(j)}$ projects out the components belonging to all partners of the j-th irreducible representation. Note that only the characters are necessary to calculate $\mathcal{P}^{(j)}$, whereas we need the matrix representations for $\mathcal{P}^{(j)}_{\lambda\kappa}$.

30. REPRESENTATIONS OF SO(3) AND SU(2)

Let us begin with $G = \mathrm{SU}(2)$. It is easy to find two representations: V_0 is the trivial representation

$$V_0 : G \;\to\; \mathrm{GL}(1, \mathbb{C}), \quad A \mapsto 1 \tag{30.1}$$

mapping each special unitary matrix A onto the complex number 1. V_0 is a one-dimensional representation. The next representation V_1 is also easy to define. The matrix representation

$$V_1 : G \;\to\; \mathrm{GL}(2, \mathbb{C}), \quad A \mapsto A \tag{30.2}$$

is defined by the identity, or the action of $G = \mathrm{SU}(2)$ on $\mathbb{C}^2$ is the standard representation defined by matrix multiplication. This representation is two-dimensional. Let us now define an $n+1$-dimensional representation V_n. As a vector space, V_n consists of homogeneous polynomials of order n with two variables z_1 and z_2. A basis of V_n is given by the $n+1$ polynomials

$$P_k(z_1, z_2) \;=\; z_1^k z_2^{n-k}, \quad 0 \le k \le n. \tag{30.3}$$

In order to describe the representation of SU(2) on G, we need to define a multiplication of a matrix $A \in \mathrm{SU}(2)$ with a polynomial. It is sufficient to define this multiplication on the basis polynomials and then extend this definition by linearity. Let $A \in \mathrm{SU}(2)$ be the matrix

$$A \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{30.4}$$

and define

$$A\left(z_1^k z_2^{n-k}\right) \;=\; A P_k(z_1, z_2) \;=\; P_k(a z_1 + c z_2, b z_1 + d z_2) \;=\; (a z_1 + c z_2)^k (b z_1 + d z_2)^{n-k}. \tag{30.5}$$

In other words, interpret the two variables z_1 and z_2 as a 2-dimensional complex vector $z = (z_1, z_2)$ and define

$$(AP)\, z \;=\; P\,(zA) \tag{30.6}$$

by matrix multiplication. We have now defined an $n+1$-dimensional complex representation of $\mathrm{SU}\,(2)$ for each $n \in \mathbb{N}$.

Theorem: The representations V_n are irreducible. Every unitary representation of $\mathrm{SU}\,(2)$ is isomorphic to one of the V_n.

Before we calculate the characters of the representations V_n, let us study the conjugacy classes of $\mathrm{SU}\,(2)$. Each unitary matrix can be diagonalized by changing the basis, therefore the conjugacy classes of $\mathrm{U}\,(n)$ are given by diagonal matrices. The special unitary matrices have determinant one and are conjugate to a diagonal matrix, but they also must be unitary, i.e., the transpose conjugate must be equal to the inverse matrix. That further restricts the choice of the diagonal elements, and we see that any special unitary matrix is conjugate to

$$e\,(t) \;=\; \begin{pmatrix} \exp\,(it) & 0 \\ 0 & \exp\,(-it) \end{pmatrix}. \tag{30.7}$$

Two matrices $e\,(s)$ and $e\,(t)$ are conjugate to each other if and only if their sum or difference is equal to an integral multiple of 2π. Therefore, the classes of $\mathrm{SU}\,(2)$ are given by the matrices $e\,(t)$, where t runs from zero to π, i.e., $0 \le t \le \pi$. In order to determine the characters of a matrix $A \in \mathrm{SU}\,(2)$, it is sufficient to define the value of the character for $t \in \mathbb{R}$.

The **characters** of the irreducible representation V_n of $\mathrm{SU}\,(2)$ have the form

$$\chi_{V_n}\,(t) \;=\; \chi_n\,(t) \;=\; \sum_{k=0}^{n} \exp\,[i\,(n-2k)\,t]\,. \tag{30.8}$$

If t is not an integral multiple of π or zero, this sum equals

$$\chi_n\,(t) \;=\; \kappa_n\,(t) \;=\; \frac{\sin\,(n+1)\,t}{\sin t}. \tag{30.9}$$

Using the addition theorem for the sin function, we get

$$\kappa_n\,(t) \;=\; \cos\; nt \;+\; \kappa_{n-1}\,(t)\cos\; t. \tag{30.10}$$

Tinkham [2] explicitly writes down the matrices for the unitary irreducible representations of $\mathrm{SU}\,(2)$ and $\mathrm{SO}\,(3)$. Tinkham [2] replaces t by $\alpha/2$ and n by $2j$ and writes the classes of the special unitary group as

$$e\,(\alpha) \;=\; \begin{pmatrix} \exp\,(i\alpha/2) & 0 \\ 0 & \exp\,(-i\alpha/2) \end{pmatrix}, \tag{30.11}$$

where α runs from 0 to 2π, therefore the characters have the form

$$\chi^{(j)}(\alpha) = \sum_{m=-j}^{j} \exp(im\alpha). \tag{30.12}$$

j is the angular momentum quantum number and can take either integral or half-integral values.

Before we discuss the representations of SO(3), we should study the relationship between SU(2) and SO(3). We have already seen (in a homework problem) that the Lie algebras su(2) and so(3) have the same dimensions. Therefore, the two groups are locally diffeomorphic, at least as manifolds. Both groups SO(3) and SU(2) can be described by three real parameters called **Euler's angles**. So what is the difference between two groups ?

Let us first study the topologies of SU(2) and SO(3). Both groups are **connected**, that means that any rotation A can be generated from infinitesimal rotations. So there is no difference here. However, let us go one step further and introduce the **fundamental group** π of the two groups SU(2) and SO(3). Let $f : \mathbb{R} \to G$ be any parameter curve into one these two groups. Geometrically speaking, there is a hole in the center of the two groups, since there are no matrices with zero determinant. A curve can either "avoid" this hole, so that we can contract it continuously to the trivial path, or it can wrap itself around the hole.

For the torus T^1, the fundamental group is $(\mathbb{Z}, +)$, since for each n we can find a path that wraps around the hole n times. It is a bit trickier to picture paths of SO(3) this way, since we need more than three dimensions. It turns out, however, that a path in SO(3) either wraps around the hole once, or it is equivalent to the trivial path. While this may sound strange, it turns out that a path that winds around twice can be pulled over in some higher dimension, and therefore is equivalent to the trivial path. For SU(2), it is even stranger: The complex variables give us an additional degree of freedom, and therefore any path in SU(2) is equivalent to the trivial path. Therefore, SU(2) is called **simply connected**. This more than just connected. To summarize, the fundamental group of a torus T^n with n holes (surface of an $n+1$-dimensional doughnut) is $\mathbb{Z}^n$, for SO(3) it is the cyclic group C_2 of order 2, for SU(2) it is the trivial group $\{1\}$.

The following theorem completely describes the connection between SU(2) and SO(3): There is an epimorphism (surjective homomorphism)

$$\pi : \ \mathrm{SU}(2) \ \to \ \mathrm{SO}(3) \tag{30.13}$$

with the kernel being

$$\ker \ \pi = \{E, -E\}, \tag{30.14}$$

where E is the identity matrix in SU(2). We can explicitly write down this epimorphism

$$\pi : \ \mathrm{SU}(2) \ \to \ \mathrm{SO}(3), \quad e(t) \ \mapsto \ R(2t), \tag{30.15}$$

with

$$e(t) = \begin{pmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{pmatrix} \quad \mapsto \quad R(2t) = \begin{pmatrix} 1 & 0 & \\ 0 & \cos\ 2t & \sin\ -2t \\ 0 & \sin\ 2t & \cos\ 2t \end{pmatrix} \tag{30.16}$$

In particular, we can see that $e(\pi) = -E$, but $R(2\pi) = E$, which proves the statement about the kernel. We also note that the classes of SO (3) are represented by $R(2t)$ with $0 \leq t < \pi$, with $R(0) = R(2\pi)$.

Suppose we have a representation V of SO (3) defined by Av for $A \in \mathrm{SO}(3)$. Then we can construct a representation of SU (2) by defining $Bv = \pi(B)v$ for a matrix $B \in \mathrm{SU}(2)$. This representation is called the **induced representation**. In this representation, $-E$ acts like the identity, that is

$$(-E)\,v \;=\; \pi(-E)\,v \;=\; v \;=\; \pi(E) \;=\; Ev. \tag{30.17}$$

On the other hand, if we have a representation of SU (2), in which $-E$ acts like the identity, then we can construct a representation of SO (3). Since our arguments are compatible with direct sums, we conclude:

Theorem: There is a one-to-one correspondence between the irreducible representations of SO (3) and those irreducible representations of SU (2), in which $-E$ acts like the identity.

The matrix $-E \in \mathrm{SU}(2)$ belongs to the class $e(\pi)$, and its character in SU (2) is given by

$$\chi_n(-E) \;=\; \chi_n(\pi) \;=\; \sum_{k=0}^{n} \exp\left[i(n-2k)\,\pi\right] \;=\; (n+1)(-1)^n\,. \tag{30.18}$$

Therefore, $-E$ acts like the identity for even n, and we conclude:

Corollary: The irreducible representations W_n of SO (3) are given by the irreducible representations V_{2n} of SU (2). The dimension of W_n is dim $W_n = 2n+1$. The character of W_n for the matrix $R(t)$ is the same as the function χ_{2n} evaluated at $e(t/2)$, and this value is

$$\chi_{W_n}(t) \;=\; \chi_{V_{2n}}(t/2) \;=\; \sum_{k=0}^{2n} \exp\left[i(n-k)\,t\right] \;=\; \frac{e^{i(n+1)t} - e^{-i(n+1)t}}{e^{it} - e^{-it}}. \tag{30.19}$$

Tinkham [2] writes this as

$$\chi^{(j)}(\alpha) \;=\; \sum_{m=-j}^{j} \exp(im\alpha) \;=\; \frac{\sin\left(j+\frac{1}{2}\right)\alpha}{\sin\frac{1}{2}\alpha}. \tag{30.20}$$

where $\alpha = t$ runs from 0 to 2π and $j = n$

Corollary: The group O (3) (consisting of all proper and improper rotations) is isomorphic to SO (3) $\times$ C_2, therefore its irreducible characters are obtained by multiplying the irreducible characters of SO (3) with either $+1$ or -1, depending on the determinant of the matrix A in O (3) whose character we want to calculate.

Theorem: Clebsch-Gordan formulas for SU (2) and SO (3):

$$V_k \otimes V_l = \bigoplus_{j=0}^{q} V_{k+l-2j} \quad \text{with} \quad q = \min\{k, l\} \quad \text{and} \tag{30.21}$$

$$W_k \otimes W_l = W_{|k-l|} \oplus W_{|k-l|+1} \oplus \ldots \oplus W_{k+l}. \tag{30.22}$$

This theorem is the justification for the vector model for the addition of angular momenta. The wave function for the system consisting of two non-interacting spins is the product of the individual wave functions, which is equal to the tensor product in mathematical terms. If $\vec{L}_1$ and $\vec{L}_2$ are the two angular momenta, then $\vec{L} = \vec{L}_1 + \vec{L}_2$. If the two vectors are parallel, then $L = L_1 + L_2$ to the maximum allowed value of L, or if they are parallel, then $L = |L_1 - L_2|$, just as the theorem states. The theorem can be proven using characters.

Physicists convention: Physicists like to make things more complicated than they need to be and talk about integral and half-integral representations, rather than even and odd representations. If we say that the representation $V(k/2)$ is identical to V_k, then the Clebsch-Gordan formula for $\mathrm{SU}(2)$ reads

$$V(a) \otimes V(b) \;\cong\; V(|a-b|) \oplus V(|a-b|+1) \oplus \ldots \oplus V(a+b). \tag{30.23}$$

We have already seen that any n-dimensional irreducible representation V_n of $\mathrm{SU}(2)$ is a complex vector space with basis functions that are homogeneous polynomials of degree n in two complex variables. Let us now find the basis functions for the representations of $\mathrm{SO}(3)$.

Let P_l be the complex vector space of homogeneous polynomials with basis functions

$$P_l(\vec{r}) \;=\; x^i y^j z^{l-i-j} \tag{30.24}$$

in three real variables $\vec{r} = (x, y, z)$ of degree l. We can define an action of $A \in \mathrm{GL}(3, \mathbb{R})$ and its subgroup $\mathrm{SO}(3)$ on P_l by stating

$$AP_l(\vec{r}) \;=\; P_l(A\vec{r}) \tag{30.25}$$

This vector space is therefore a representation of $\mathrm{SO}(3)$, but it is reducible for $n > 1$, for example the space generated by the function $|\vec{r}|^2 = x^2 + y^2 + z^2$ is a one-dimensional G-invariant subspace of P_2. Let us define the space of **harmonic polynomials** of degree l as

$$\mathcal{H}_l \;=\; \{f \in P_l \,|\, \Delta\, f = 0\}, \tag{30.26}$$

where Δ is the Laplace-operator. Restricting $\mathcal{H}_l$ to the surface of the sphere S^2 yields the spherical harmonics of degree l. I remind you that a basis of the spherical harmonics of degree l is given by

$$Y_m(\phi, \theta) = \exp(im\phi)\, P_l^m(\cos\theta), \quad \text{where} \tag{30.27}$$

$$P_l(t) = \frac{1}{2^l l!} \frac{d^l}{dt^l} \left(t^2 - 1\right)^l \quad \text{and} \tag{30.28}$$

$$P_l^m(t) = \left(1 - t^2\right)^{m/2} \frac{d^m}{dt^m} P_l(t). \tag{30.29}$$

The functions P_l are called **Legendre functions**, and the P_l^m are the **associated Legendre functions**. With the proper normalization, we can obtain the normalized spherical harmonics $Y_l^m(\phi, \theta)$ for $-l \leq m \leq l$.

We note that dim $P_l = \frac{1}{2}(l+1)(l+2)$ and dim $\mathcal{H}_l = 2l+1$. We also note that the Laplace-operator commutes with the action of SO (3) on P_l, therefore $\mathcal{H}_l$ forms an SO (3)-invariant subspace of P_l. Now we have the connection between special functions and symmetry properties:

Theorem: The space $\mathcal{H}_l$ of harmonic polynomials of degree l is an irreducible SO (3)-module W_l of dimension $2l+1$, and every irreducible representation of SO (3) can be expressed in this way. Therefore, the spherical harmonics of degree l are the basis functions of W_l.

Exercises:

(X1) Using similar techniques, find the irreducible representations of U (2) and O (3).

(X2) Prove the Clebsch-Gordan formula for SU (2). Hint: It is sufficient to prove the formula for characters.

(X3) Derive a similar Clebsch-Gordan formula for the group U (2).

31. PROJECTIVE REPRESENTATIONS OF THE ROTATION GROUP

We have defined a complex matrix representation of a (Lie) group G as a (continuous) homomorphism $G \to \mathrm{GL}(n, \mathbb{C})$. However, this formalism is not really adequate for quantum mechanics because of the undetermined phase factor of the quantum-mechanical wave function. If two quantum-mechanical have wave functions that are only different by a phase, then these two states are really identical, since all observable quantities are the same. Therefore, a quantum-mechanical state is really not a single vector in Hilbert space, but rather an equivalence class of vectors identical up to a phase or, if you wish, a line through the origin. If we also take into account that the wave functions have to be normalized to one, then a quantum mechanical state is an equivalence class $\lambda\psi$ with $\lambda \in S_1$ and $|\psi| = 1$.

Since two wave functions are considered identical, if their only difference is a complex phase, two representations Γ_1 and Γ_2 should also be considered identical, if for all $g \in G$ we have $\Gamma_1(g) = \exp(i\phi)\,\Gamma_2(g)$ with some fixed ϕ independent of g. This leads to the following

Definition: A projective representation of a Lie group G is a continuous homomorphism

$$G \;\to\; \mathrm{PGL}(n, \mathbb{C}) \;=\; \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^* \;=\; \mathrm{SL}(n, \mathbb{C})/C_n \tag{31.1}$$

where C_n is the cyclic group of order n consisting of the n-th roots of unity. In other words, since the phases are irrelevant, we factor them out.

All that counts in quantum mechanics are projective representations. The representations we have discussed in detail over the past few weeks are completely irrelevant for quantum mechanics. Projective representations of groups were introduced by Schur (1911) and have become a fashionable area of study in mathematics in recent years. Treatments exist [18–20], but I haven't found one that is readable. We therefore will leave this paragraph after the following [10]

Theorem: The projective representations of SO (3) are given up to conjugation by the representations of SU (2) which have the form either

$$\sum_n k_n V_{2n} \quad \text{or} \quad \sum_n k_n V_{2n+1}, \quad k_n \in \mathbb{N}_0. \tag{31.2}$$

The first kind are called **even** and the second kind are called **odd**.

The physical significance of this mathematical theorem is the following: A quantum mechanical wave function for a particle consists of a sum of components, which either belong to even or odd representations of the group SU (2). No particle can have a wave function with both odd and even components. Therefore, all quantum mechanical particles fall into two classes: **Fermions** (with wave functions belonging to odd representations of SU (2)) and **bosons** (with wave functions belonging to even representations). This classification into bosons and fermions is a mathematical conclusion we draw from the fact that the quantum-mechanical phase cannot be determined. If this phase could actually be measured, there could be no fermions. From the experimental fact that fermions (particles with half-integral spin) exist, we can conclude using group-theoretical arguments that the phase factor cannot be measured.

Let me state this again: If you were told in your quantum mechanics course about rotations by 4π and never could figure it out, don't worry: You were fooled with a simplification. There are no rotations by 4π in nature. We all know that a rotation by 2π is the identity. However, because of the undetermined quantum mechanical phase, we need the representations of the group SU (2), which looks like rotations in SO (3), where the angle runs from 0 to 4π.

There is a more general version of this theorem (if I remember correctly), which states that the projective representations of any Lie group are given by the representations of the universal cover group (used here in a topological sense). Therefore, if a group is simply connected, such as $\mathbb{R}^n$, then the projective representations and the representations are the same. We conclude that we do not have to worry about the phase factor for the translation group. I have no clue, however, about the projective represenations of finite (or not-connected) groups. Let me add one last statement for the experts.

Let G be a connected group. Then we can find a topological cover which is simply connected. This is called the **universal cover** $\tilde{G}$. The projection from $\tilde{G}$ into G is called p. If $\phi : G \to \mathrm{SU}(n)/C_n$ is an n-dimensional projective representation by special unitary matrices, then there is unique **lifting** $\tilde{\phi}$ which makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\phi}} & \mathrm{SU}(n) \\ p\downarrow & & \downarrow \\ G & \xrightarrow{\phi} & \mathrm{SU}(n)/C_n \end{array} \tag{31.3}$$

32. DOUBLE GROUPS AND CRYSTAL SYMMETRY

We have seen in the previous chapters that the irreducible representations of SO (3) are given by those irreducible representations of SU (2), where $-E$ acts like the identity. The irreducible characters of O (3) and SO (3) are the same for proper rotations, and for an improper rotation we multiply the character of the corresponding improper rotation by -1. The 32 point groups are subgroups of O(3), therefore the characters of the point groups are special cases with discrete sets of parameters. These are the characters that are tabulated for the 32 point groups in appendix B of the book by Tinkham [2], and these are used for elementary excitations in solids that do not require the measurement of a quantum-mechanical phase. For a phonon, for example, these representations are adequate.

For electronic states in solids, however, we may have to use the projective representations of the 32 point groups (since we cannot measure the phase), at least if we want to take into account the effects of the spin. The projective representations of the point groups are called double-group representations. A (proper) representation of a point group corresponds to the representation of a subgroup of SU(2), where $-E$ acts like the identity. In physics textbooks, the notation $\bar{E}$ is normally used instead of $-E$. In a projective representation, on the other hand, $\bar{E}$ does not have to act like the identity, but it can also act like multiplication by -1.

It might seem that there **should be** twice as many projective (or double-group) representations than (proper) representations, since $\bar{E}$ can either act as multiplication by 1 (identity) or -1. However, this is very wrong !!! The characters of the 32 double-groups (for the 32 point groups) are given by Koster *et al.* [21] and also by Elliot [22] for some double space groups relevant for spin-orbit coupling in solids.

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